FULL NAME: key SECTION: ______ Exam 1

MATH 308, Differential Equations Dr. Adam Larios No calculators

Answers without full, proper justification will not receive full credit.

1. **(8 points)** Solve the following initial value problem.

$$\begin{cases} \dfrac{dy}{dt} = y^{-2}\cos(t), \\ y(0) = 2. \end{cases}$$

Separate:

$$y^2\,dy = \cos(t)\,dt$$

Integrate:

$$\int y^2\,dy = \int \cos(t)\,dt$$

$$\Rightarrow \quad \frac{1}{3}y^3 + C_1 = \sin(t) + C_2$$

$$\Rightarrow \quad y^3 = 3\sin(t) + C_3$$

$$C_3 = 3(C_2 - C_1)$$

does not distribute

$$y(t) = (3\sin t + C_3)^{1/3}$$

Plug in initial condition:

$$2 = y(0) = (3\sin 0 + C_3)^{1/3}$$

$$\Rightarrow \quad 2 = C_3^{1/3} \Rightarrow C_3 = 2^3 = 8$$

$$\Rightarrow \quad \boxed{y(t) = (3\sin t + 8)^{1/3}}$$

2. **(10 points)** Solve the following equation up to an arbitrary constant c.

$$\frac{dy}{dt} + \frac{4}{t}y = \frac{1}{t^3} - 1$$

Not separable, but linear, so use integrating factor:

$$\mu(t) = e^{\int \frac{4}{t}dt} = e^{4\ln|t|} = e^{\ln|t|^4} = |t|^4 = t^4$$

Multiply by μ:

$$\underbrace{t^4\frac{dy}{dt} + 4t^3y}_{(t^4y)'} = t - t^4$$

$$(t^4y)' = t - t^4$$

$$\Rightarrow \quad t^4 y = \frac{1}{2}t^2 - \frac{1}{5}t^5 + C$$

$$\Rightarrow \quad y(t) = \frac{1}{2}\frac{1}{t^2} - \frac{1}{5}t + \frac{C}{t^4}$$

Cannot absorb t^4 into constant, since then it would not be constant.

3. **(6 points)** Consider the following equation.

$$y' + \ln(t)\, y = \frac{1}{4-t^2}$$

- $\ln(t)$ continuous on $(0, \infty)$,
- $\frac{1}{4-t^2}$ continous on $(-\infty, -2)$, $(-2, 2)$ and $(2, \infty)$.

(a) According to the theorems we learned in this class, what is the largest interval of time for which a solution satisfying $y(1) = 5$ is guaranteed to exist and be unique?

$t_0 = 1$ is in both $(0, \infty)$ and $(-2, 2)$, so largest existence interval is their intersection, namely $(0, 2)$

(b) According to the theorems we learned in this class, what is the largest interval of time for which a solution satisfying $y(99) = 17$ is guaranteed to exist and be unique?

$t_0 = 99$ is in both $(0, \infty)$ and $(2, \infty)$ so largest of existence interval is $(2, \infty)$

4. **(8 points)** Consider the initial value problem given by:

$$\frac{dy}{dt} = (y-2)^{1/3} t^2, \qquad y(1) = y_0,$$

where **we think of y_0 as a given, fixed number**. Is this problem guaranteed to have a unique solution by the theorems we learned in class? Is there anything that could cause it to not have a unique solution? **Do not solve the equation.**

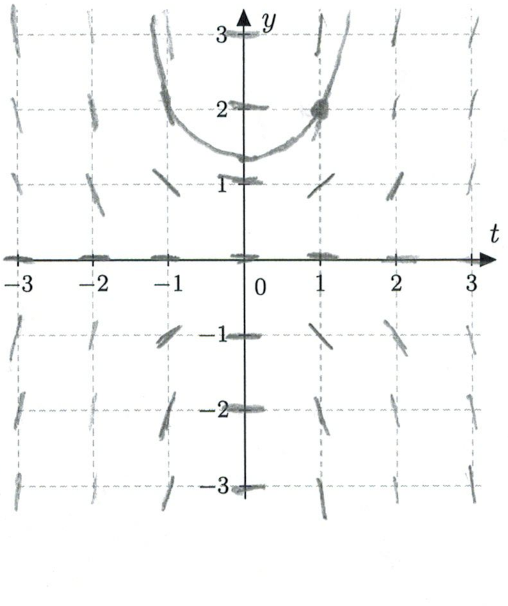
Let $f(t,y) = (y-2)^{1/3} t^2$.

Notice $f(t,y)$ is continuous everywhere,

but $\frac{\partial f}{\partial y} = \frac{1}{3}(y-2)^{-2/3} t^2$ is not continuous at $y = 2$.

Thus, if $y_0 \neq 2$ solution is unique, but if $y_0 = 2$, solution might not be unique.

5. **(8 points)** Plot a direction field as completely as possible for the following equation, and sketch the integral curve which passes through the point $(1, 2)$.



$\boxed{y' = ty}$

No need to solve equation.

6. **(12 points)** We saw in class populations with carrying capacities. Suppose a population $P(t)$ has a carrying capacity of 200, but it also has a threshold, where the population starts to drop if it is below 100. This can be modeled by

$$P' = rP(1 - \tfrac{P}{200})(\tfrac{P}{100} - 1)$$

where $r > 0$ is the intrinsic growth rate. **Don't try to solve the equation.**

(a) Identify the equilibrium points of this model. Set $P' = 0$. Then $0 = rP(1-\frac{P}{200})(\frac{P}{100}-1)$
So $P=0$, $P=200$ and $P=100$ are equilibria.

(b) Draw the corresponding phase line, and identify which equilibria are stable (S), and which are unstable (U).

Look at sign of P'
That is, sign of
$rP(1-\frac{P}{200})(\frac{P}{100}-1)$

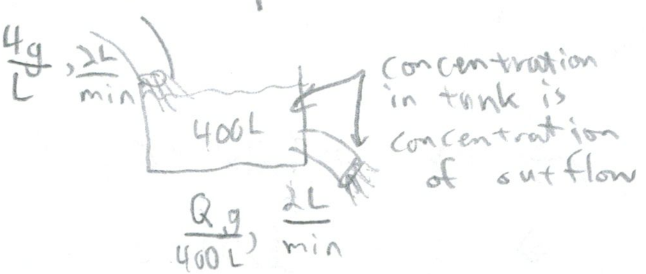
$$\longleftrightarrow \; >>> \underset{P=0}{\overset{S}{\bullet}} <<< \underset{P=100}{\overset{U}{\bullet}} >>> \underset{P=200}{\overset{S}{\bullet}} <<<$$

(c) Suppose the initial population is $P(0) = P_0 = 150$. Find $\lim_{t\to\infty} P(t)$.

Since $\frac{dP}{dt} > 0$ in the interval $(100, 200)$, If $P_0 = 150$,
$P(t)$ will tend toward $P = 200$ as $t \to \infty$

7. **(12 points)** A tank initially contains **400L** of pure water. Water with a concentration of **4g/L** of salt is then pumped into the tank at the rate of **2L/min**, and the well-stirred mixture leaves at the same rate. How long does it take for the concentration of salt in the tank to become **1g/L**? **(You do not need to find the decimal value.)**

Draw a picture (with labels)



Let $Q(t)$ = amount of salt at time t. Then Concentration in $\frac{Q(t)}{400L}$

$$\frac{dQ}{dt} = \left(\text{rate salt in}\right) - \left(\text{rate salt out}\right) = \left(\frac{4g}{L}\right)\left(2\frac{L}{min}\right) - \left(\frac{Q\,g}{400L}\right)\left(\frac{2L}{min}\right)$$

$\Rightarrow \frac{dQ}{dt} = 8 - \frac{1}{200}Q$ ← separable, but can also use integrating factor

$\Rightarrow \frac{1}{8-\frac{1}{200}Q}\,dQ = dt$

$\Rightarrow -200\ln\left|8 - \frac{1}{200}Q\right| = t + c_1$

$\Rightarrow 8 - \frac{1}{200}Q = c_2 e^{\frac{-t}{200}}$

Initial pure: $Q(0) = 0$

$\Rightarrow 8 - \frac{1}{200}\cdot 0 = c_2 e^0 \Rightarrow c_2 = 8$

$\Rightarrow \boxed{Q = 1600(1 - e^{-t/200})}$

$(1\frac{g}{L})(400L) = 400g$

Set $Q = 400$.

$\Rightarrow 400 = 1600(1 - e^{-t/200})$

$\Rightarrow \boxed{t = -200\ln\left(\frac{3}{4}\right)}$

8. **(4 points)** Consider the equation: $\underbrace{(\cos(x) + 3y^2)}_{M} \, dx + \underbrace{6xy}_{N} \, dy = 0.$
Is this equation exact? Justify your answer. **Do not solve the equation.**

$M_y = 6y$ $\qquad$ $M_y = N_x$, so it is exact.

$N_x = 6y$

9. **(10 points)** Consider the equation: $-\cos^2(y) \, dx + (2x\cos(y)\sin(y) + y) \, dy = 0.$
This equation is exact. (Don't check exactness.) Find its solution up to a constant.

Set $\begin{cases} f_x = M = -\cos^2 y & \text{---} \ 1 \\ f_y = N = 2x\cos y \sin y + y & \text{---} \ 2 \end{cases}$

From 1: $f = -x\cos^2 y + g(y)$

$\Rightarrow 2x\cos y \sin y + y = f_y = 2x\cos y\sin y + g'(y)$

from 2

Thus, $y = g'(y) \Rightarrow g(y) = \frac{1}{2}y^2$

Thus, $f(x,y) = -x\cos^2 y + \frac{1}{2}y^2$

So solutions are given by:

$$\boxed{-x\cos^2 y + \frac{1}{2}y^2 = C}$$

10. **(12 points)** Consider the equation: $x^2\frac{dy}{dx} = xy + y^2.$
Solve the equation up to a constant c. (**HINT:** Use the substitution $v = \dfrac{y}{x}$, or $y = xv$.)

$y = x \cdot v \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

Divide original equation by x^2:

$\frac{dy}{dx} = \frac{y}{x} + \left(\frac{y}{x}\right)^2$

Substitute:

$\underbrace{\cancel{v} + x\frac{dv}{dx}}_{\frac{dy}{dx}} = \cancel{v} + v^2$

$\Rightarrow x\frac{dv}{dx} = v^2$

Separable: $v^{-2}\,dv = x^{-1}\,dx$

$\int v^{-2}\,dv = x^{-1}\,dx$

$\Rightarrow -v^{-1} = \ln|x| + c$

$\Rightarrow \frac{1}{v} = -\ln|x| - c$

$\Rightarrow v = \frac{1}{-\ln|x| - c}$

Substitute back:

$\frac{y}{x} = \frac{1}{-\ln|x| - c}$

$\Rightarrow$ $\boxed{y = \frac{x}{-\ln|x| - c}}$

11. **(10 points)** Consider the problem

$$\begin{cases} y' = ty - 2, \\ y(3) = 2. \end{cases}$$

Use the **forward Euler** method with step size $h = 0.5$ to approximate $y(4)$.

Hint: Think about what t_0 and y_0 are before you begin.

Forward Euler: Let $f(t,y) = ty - 2$.

$$\frac{y_{n+1} - y_n}{h} = f(t_n, y_n).$$

or

$$y_{n+1} = y_n + h f(t_n, y_n) = y_n + h(t_n y_n - 2)$$

Let $y_0 = 2$. y_0 corresponds to $y(3)$.

We take step size $h = 0.5$, so need two more steps to approximate $y(4)$.

$$\boxed{\begin{aligned} y_0 &= y(3) \\ y_1 &\approx y(3.5) \\ y_2 &\approx y(4) \end{aligned}}$$

Calculation

$y_0 = 2$, $t_0 = 3$

First step:

$$y_1 = y_0 + h(t_0 y_0 - 2)$$

$$= 2 + 0.5(3 \cdot 2 - 2) = 2 + 0.5(4) = 2 + 2 = 4$$

So $y_1 = 4$, and clearly $t_1 = t_0 + h = 3 + 0.5 = 3.5$

Second step

$$y_2 = y_1 + h(t_1 y_1 - 2)$$

$$= 4 + 0.5((3.5)(4) - 2) = 4 + 0.5(14 - 2)$$

$$= 4 + 0.5(12) = 4 + 6 = 10$$

$$\boxed{y(4) \approx 10}$$

(Note: You probably don't need the whole page for this; this is just extra space.)