FULL NAME: KEY SECTION: ________ Exam 2

MATH 221, Differential Equations Dr. Adam Larios No calculators

Answers without full, proper justification will not receive full credit.

| Method of variation of parameters for $ay'' + by' + cy = g$ : |
|---|
| $y_p = y_1u_1 + y_2u_2,\qquad u_1' = \frac{gy_2}{W},\qquad u_2' = \frac{-gy_1}{W},\qquad W = y_1y_2' - y_1'y_2 =$ Wronskian |

1. **(6 points)** Write down the form of the **general** solution to the following equation using the method of undetermined coefficients. **You are not asked to solve for the constants!**

$$y'' + 6y' + 9y = 5e^{-t} + 7e^{2t}\sin(3t)$$

Characteristic: $r^2+6r+9=0 \Rightarrow (r+3)^2=0 \Rightarrow r=-3$, repeated root

$$y = \underbrace{c_1e^{-3t} + c_2te^{-3t}}_{y_h} + \underbrace{Ae^{-t} + Be^{2t}\sin(3t) + Ce^{2t}\cos(3t)}_{y_p}$$

2. **(10 points)** Find a **particular** solution solution to the following equation using the method of undetermined coefficients.

$$y'' + 4y' + 2y = 8t + 6$$

$y_p = At+B$

$y_p' = A$

$y_p'' = 0$

plug in:

$0 + 4(A) + 2(At+B) = 8t+6$

$\Rightarrow (2A)t + (4A+2B) = 8t+6$

$\Rightarrow \begin{cases} 2A = 8 \Rightarrow A=4 \\ 4A+2B = 6 \Rightarrow 4(4)+2B=6 \Rightarrow 16+2B=6 \Rightarrow B=-5 \end{cases}$

$\Rightarrow \boxed{y_p(t) = 4t-5}$

3. **(8 points)** Find the **general** solution of the following equation

$$y'' + y' - 2y = 2e^t.$$

Note: Be a little careful here!

characteristic: $r^2+r-2=0$

$\Rightarrow (r+2)(r-1)=0$

$\Rightarrow r=-2$ or $1$

$\Rightarrow y_h = c_1e^{-2t} + c_2e^t$

resonance with right-hand side, so guess $y_p = Ate^t$

$\Rightarrow y_p' = Ae^t + Ate^t = Ae^t(1+t)$

$y_p'' = Ae^t + Ae^t(1+t) = Ae^t(2+t)$

Plug in:

$Ae^t(2+t) + Ae^t(1+t) - 2Ate^t = 2e^t$

$\Rightarrow (2A+A)e^t = 2e^t$

$\Rightarrow A = \frac{2}{3}$

$\Rightarrow \boxed{y = c_1e^{-2t} + c_2e^t + \frac{2}{3}te^t}$

9. **(12 points)** Find the solution of the following IVP. **Show all your steps.**

$$y'' + 5y' + 6y = 24e^t, \qquad y(0) = 2, \qquad y'(0) = 7.$$

char: $r^2 + 5r + 6 = 0$

$\Rightarrow (r+2)(r+3) = 0$

$\Rightarrow y_h = c_1 e^{-2t} + c_2 e^{-3t}$

$y_p = Ae^t$ (guess)

$y_p' = Ae^t$

$y_p'' = Ae^t$

$\Rightarrow Ae^t + 5Ae^t + 6Ae^t = 24e^t$

$\Rightarrow 12A = 24 \Rightarrow A = 2$

$\Rightarrow y_p = 2e^t$

$\Rightarrow y = c_1 e^{-2t} + c_2 e^{-3t} + 2e^t$

$2 = y(0) = c_1 e^0 + c_2 e^0 + 2e^0$

$\Rightarrow 2 = c_1 + c_2 + 2$

$\Rightarrow c_1 = -c_2$

$y' = -2c_1 e^{-2t} - 3c_2 e^{-3t} + 2e^t$

$7 = y'(0) = -2c_1 - 3c_2 + 2$

$\Rightarrow 5 = -2c_1 - 3c_2 = -2c_1 + 3c_1 = c_1$

$\Rightarrow c_2 = -c_1 = -5$

$\Rightarrow$

$$\boxed{y = -5e^{-2t} + 5e^{-3t} + 2e^t}$$

10. **(12 points)** Find a particular solution of the equation: $\quad y'' + y = \dfrac{1}{\sin(x)}$.

Hint: If you get a funny integral somewhere, try basic substitution. Also, note that it is easier to work with cos and sin rather than tan, cot, sec, csc.

$r^2 + 1 = 0 \Rightarrow r = \pm i \Rightarrow y_h = c_1 \underset{y_1}{\cos x} + c_2 \underset{y_2}{\sin x}$

$W[\overset{y_1}{\cos x}, \overset{y_2}{\sin x}] = (\cos x)(\sin x)' - (\cos x)'(\sin x) = \cos^2 x + \sin^2 x = 1$

From box on page 1:

$u_1' = \dfrac{y_2 g}{W} = \dfrac{\sin x \cdot \left(\frac{1}{\sin x}\right)}{1} = 1 \Rightarrow u_1 = x$

$u_2' = \dfrac{-y_1 g}{W} = \dfrac{-\cos x \cdot \left(\frac{1}{\sin x}\right)}{1} = \dfrac{-\cos x}{\sin x} \Rightarrow u_2 = -\int \dfrac{\cos x}{\sin x}\,dx = \int -\dfrac{1}{u}\,du = -\ln|u|$

$u = \sin x$

$du = \cos x\,dx$

$= -\ln|\sin x|$

$\Rightarrow$

$y_p = u_1 y_1 + u_2 y_2 = x\cos x - \ln|\sin x| \cdot \sin x$

6. **(6 points)** Show that $y_1 = e^t$ and $y_2 = te^t$ are linearly independent.

Check wronskian (formula on page 1)

$$W = y_1 y_2' - y_1' y_2 = (e^t)(te^t)' - (e^t)'(te^t)$$
$$= (e^t)(te^t + e^t) - (e^t)(te^t) = e^{2t} \neq 0,$$ so $y_1$ & $y_2$ are lin. indep.

7. **(8 points)** Consider the following equation for $t > 0$:

$$y'' - \frac{1}{t}y' + t^2 y = 0.$$

By guessing a solution of the form $y(t) = \sin(rt^2)$ and solving for $r > 0$, find a non-zero solution.

$$y = \sin(rt^2)$$
$$y' = \cos(rt^2) \cdot 2rt$$
$$y'' = 2r\cos(rt^2) - \sin(rt^2) \cdot 4r^2t^2$$

Plug in:

$$0 = y'' - \frac{1}{t}y' + t^2 y = \left[\cancel{2r\cos(rt^2)} - \sin(rt^2) \cdot 4r^2t^2\right] - \cancel{\frac{1}{t}\left[\cos(rt^2) \cdot 2rt\right]} + t^2\sin(rt^2)$$
$$= (-4r^2+1)t^2\sin(rt^2)$$
$$\Rightarrow -4r^2+1 = 0 \Rightarrow r = \pm\sqrt{\tfrac{1}{4}} = \pm\tfrac{1}{2}. \quad r>0 \Rightarrow r = \tfrac{1}{2}$$

$$\boxed{y = \sin(\tfrac{1}{2}t^2)}$$

8. **(8 points)** Suppose $y_1$ and $y_2$ are two solutions to the equation $ay'' + by' + cy = 0$. Show that $y = 5y_1 + 3y_2$ is also a solution.

Just plug in and see that you get zero:

$$a(5y_1 + 3y_2)'' + b(5y_1 + 3y_2)' + c(5y_1 + 3y_2)$$
$$= 5ay_1'' + 3ay_2'' + 5by_1' + 3by_2' + 5cy_1 + 3cy_2$$
$$= 5(ay_1'' + by_1' + cy_1) + 3(ay_2'' + by_2' + cy_2)$$
$$= 5 \cdot 0 + 3 \cdot 0$$
$$= 0$$ ✓ This is verifying the principal of superposition.

4. **(10 points)**

(a) Write the function $z(t) = e^{(2+3i)t}$ in the form $a(t) + b(t)i$ where $a(t)$ and $b(t)$ are real, and $i = \sqrt{-1}$.

$$e^{(2+3i)t} \underset{\text{exp. rules}}{=} e^{2t} \cdot e^{3ti} \underset{\text{Euler's formula}}{=} e^{2t}\left(\cos(3t) + i\sin(3t)\right)$$

$$= e^{2t}\cos(3t) + i e^{2t}\sin(3t)$$

(b) Suppose the motion of a certain mechanical system is governed by the equation

$$y'' - 4y' + 13y = 0.$$

Does the system have oscillations? **Justify your answer using mathematics.**

char: $r^2 - 4r + 13 = 0$

$$\Rightarrow r = \frac{4 \pm \sqrt{4^2 - 4 \cdot 1 \cdot 13}}{2}$$

$= 2 \pm 3i$ ← non-zero imaginary part, so there will be oscillations. See

(c) Describe what happens to the system as $t \to \infty$. Please keep answers very **brief** and **mathematical**.

General solution is $c_1 e^{2t}\cos(3t) + c_2 e^{2t}\sin(3t)$.

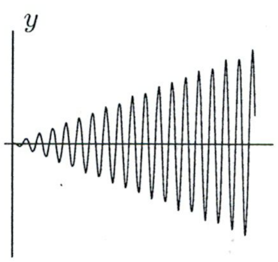
Thus, as $t \to \infty$, solution oscillations, and amplitude grows exponentially.

5. **(8 points)** Write the letter of the equation next to the graph that best represents a solution to it. **There is no need to solve the equation**, just think about how the equation behaves.

(A) $y'' + 16y = 10$
(B) $y'' + 16y = 5\cos(3t)$
(C) $y'' + 16y = \frac{1}{2}\cos(4t)$

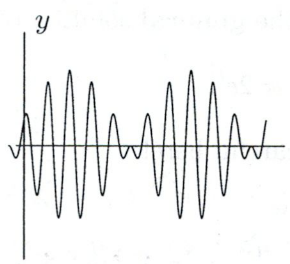
$r^2 + 16 = 0$
$\Rightarrow r = \pm 4i$

(D) $y'' + 16y = -10$
(E) $y'' + 16y = 4e^{-t}$
(F) $y'' + 2y' + 16y = 0$



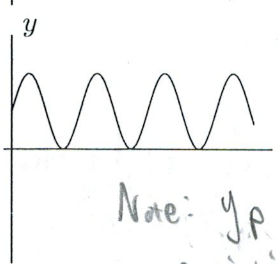

C

resonant forcing



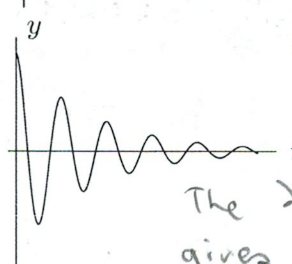

B

$\cos(3t)$ gives a beat frequency with $\sin(4t)$ & $\cos(4t)$.




A

Note: $y_p$ must be positive to match w/ 10




F

The $2y'$ term gives damping

11. **(12 points)** Consider the equation (for $t > 0$), given by

$$ty'' + (1 - 2t)y' + (t - 1)y = 0$$

One solution is given by $y_1(t) = e^t$. Find another solution and give the **general solution**.

Use reduction of order. Set:

$$y_2 = v y_1 = v e^t$$

$$\Rightarrow y_2' = v' e^t + v e^t$$

$$\Rightarrow y_2'' = v'' e^t + 2v' e^t + v e^t$$

Thus,

sum

$$\begin{aligned} (t-1) y_2 &= \cancel{(t-1)e^t v} \\ (1-2t) y_2' &= (1-2t) e^t v' + \cancel{(1-2t) e^t v} \\ t y_2'' &= t e^t v'' + 2te^t v' + \cancel{t e^t v} \\ \hline 0 &= t e^t v'' + e^t v' \end{aligned}$$

sum

since $y_2$ is assumed to be a solution

$\Rightarrow \quad 0 = tv'' + v'$. Let $w = v'$. Then $0 = tw' + w$.

separable and linear

Solve, for example, using integrating factor

$$w' + \frac{1}{t} w = 0 \Rightarrow \mu = e^{\int \frac{1}{t} dt} = e^{\ln|t|} = |t| \Rightarrow (w|t|)' = 0 \Rightarrow w = \frac{c_1}{t}$$

or separable:

$$-w = t\frac{dw}{dt} \Rightarrow \frac{-1}{t} dt = \frac{1}{w} dw \Rightarrow -\ln|t| = \ln|w| + c_1 \quad \ln\left|\frac{1}{t}\right| = \ln|w| + c_2 \Rightarrow w = \frac{c_3}{t}$$

So $v' = w = \frac{c_4}{t} \Rightarrow v = c_4 \ln|t|$. Take $v = \ln|t|$ for example.



Thus, $y_2 = e^t \ln|t|$