FULL NAME: Key SECTION: ________ Exam 3

MATH 221, Differential Equations Dr. Adam Larios No calculators

Answers without full, proper justification will not receive full credit.

Here f and g are functions that have well-defined Laplace transforms, a, b, c are real constants, $\mathcal{L}\{f\}(s) = F(s)$ is the Laplace transform of f, and $\mathcal{L}\{g\}(s) = G(s)$ is the Laplace transform of g.

Table of Laplace Transforms:

$f(t)$	$F(s) = \mathcal{L}\{f\}(s) = \int_0^\infty f(t)e^{-st}\,dt$
1	$\frac{1}{s}$
$t^n, n = 1, 2, 3, \ldots$	$\frac{n!}{s^{n+1}}$
$\cos(bt)$	$\frac{s}{s^2+b^2}$
$\sin(bt)$	$\frac{b}{s^2+b^2}$
$\sinh(bt) = \frac{e^{bt}-e^{-bt}}{2}$	$\frac{b}{s^2-b^2}$
$\cosh(bt) = \frac{e^{bt}+e^{-bt}}{2}$	$\frac{s}{s^2-b^2}$
e^{at}	$\frac{1}{s-a}$
$e^{at}f(t)$	$F(s-a) = \mathcal{L}\{f(t)\}(s-a)$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$
$f'(t)$	$sF(s) - f(0) = s\mathcal{L}\{f\}(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0) = s^2\mathcal{L}\{f\}(s) - sf(0) - f'(0)$
$f^{(n)}(t)$ n^{th} derivative of $f(t)$	$s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0)$
$tf(t)$	$-\frac{d}{ds}[F(s)] = -\frac{d}{ds}[\mathcal{L}\{f\}(s)]$
$t^nf(t)$	$(-1)^n\frac{d^n}{ds^n}[F(s)] = (-1)^n\frac{d^n}{ds^n}[\mathcal{L}\{f\}(s)]$
$u_c(t) = u(t-c)$	$\frac{1}{s}e^{-sc}$
$f(t-c)u_c(t) = f(t-c)u(t-c)$	$e^{-sc}F(s) = e^{-sc}\mathcal{L}\{f(t)\}(s)$
$\delta_c(t) = \delta(t-c)$	e^{-sc}
$t^ne^{at}, n = 1, 2, 3, \ldots$	$\frac{n!}{(s-a)^{n+1}}$
$af(t) + bg(t)$	$a\mathcal{L}\{f\} + b\mathcal{L}\{g\}$
$(f * g)(t) := \int_0^t f(t-\tau)g(\tau)\,d\tau$	$\mathcal{L}\{f\}\mathcal{L}\{g\} = F(s)G(s)$

1. **(12 points)** Solve the following initial value problem.

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}\mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$$

Find eigenvalues:

$$0 = \det\begin{pmatrix} 5-\lambda & -1 \\ 3 & 1-\lambda \end{pmatrix} = (5-\lambda)(1-\lambda) - (-1)(3)$$

$$\Rightarrow 0 = \lambda^2 - \lambda - 5\lambda + 5 + 3 = \lambda^2 - 6\lambda + 8$$

$$\Rightarrow 0 = (\lambda - 2)(\lambda - 4) \Rightarrow \lambda = 2 \text{ or } 4$$

Find eigenvectors:

$\lambda = 2$

$$\begin{pmatrix} 5-2 & -1 \\ 3 & 1-2 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} 3x_1 - y_1 = 0 \\ 3x_1 - y_1 = 0 \end{cases}$$

Choose $x_1 = 1$, then $y_1 = 3$

$$\vec{x}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$\lambda = 4$

$$\begin{pmatrix} 5-4 & -1 \\ 3 & 1-4 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} x_2 - y_2 = 0 \\ 3x_2 - 3y_2 = 0 \end{cases}$$

choose, e.g., $y_2 = 1$. Then $x_2 = 1$.

$$\vec{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

General solution:

$$\vec{x}(t) = c_1 e^{2t}\begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 e^{4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Initial condition:

$$\begin{pmatrix} 0 \\ 4 \end{pmatrix} = \vec{x}(0) = c_1\begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{cases} 0 = c_1 + c_2 \\ 4 = 3c_1 + c_2 \end{cases}$$

$$\Rightarrow c_1 = -c_2 \Rightarrow c_1 = 2 \Rightarrow c_2 = -$$

$$\Rightarrow \boxed{\vec{x}(t) = 2e^{2t}\begin{pmatrix} 1 \\ 3 \end{pmatrix} - 2e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

2. **(10 points)** Compute the inverse Laplace transform, $\mathcal{L}^{-1}$, of

$$F(s) = \frac{1}{s^2 + 2s + 5} + \frac{1}{s^2 + 10s + 21}$$

Complete the square

$$F(s) = \frac{1}{2}\frac{2}{(s+1)^2 + 2^2} + \frac{1}{(s+3)(s+7)}$$

Partial fractions:

$$\frac{1}{(s+3)(s+7)} = \frac{A}{s+3} + \frac{B}{s+7} \Rightarrow 1 = A(s+7) + B(s+3).$$

choose $s = -7 \Rightarrow B = -1/4$

choose $s = -3 \Rightarrow A = 1/4$

$$= \frac{1/4}{s+3} + \frac{-1/4}{s+7}$$

Thus, $\boxed{\mathcal{L}^{-1}[F(s)] = \frac{1}{2}e^{-t}\sin(2t) + \frac{1}{4}e^{-3t} - \frac{1}{4}e^{-7t}}$

3. **(3 points)** Does the Laplace transform of $f(t) = e^{(t^4)}$ exist? Briefly state why or why not.

No, since the function grows faster than exponentially.

4. **(12 points)** Consider a 2×2 linear system $\frac{d}{dt}\vec{\mathbf{x}} = A\vec{\mathbf{x}}$ where A is a constant matrix with eigenvalues λ_1 and λ_2. Match each pair of eigenvalues [left] with the letter of the corresponding behavior near the origin [right].

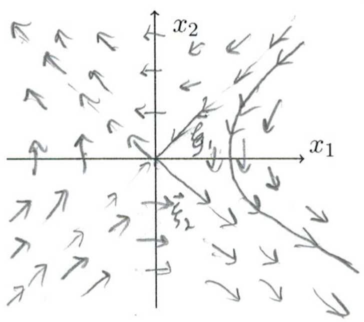
$\lambda_1 = 2,$	$\lambda_2 = 3$	: C	(A) Center
$\lambda_1 = 2+3i,$	$\lambda_2 = 2-3i$	: F	(B) Saddle Node
$\lambda_1 = -2,$	$\lambda_2 = 3$	: B	(C) Non-spiral Source
$\lambda_1 = -2+3i,$	$\lambda_2 = -2-3i$	: E	(D) Non-spiral Sink
$\lambda_1 = -2,$	$\lambda_2 = -3$	: D	(E) Spiral Sink
$\lambda_1 = 2i,$	$\lambda_2 = -2i$	: A	(F) Spiral Source

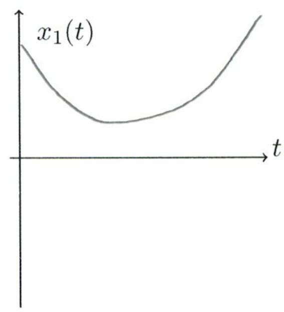
It may help to remember: $e^{(a+bi)t} = e^{at}\left(\cos(bt) + i\sin(bt)\right)$

5. **(10 points)** Consider the ODE system given by

$$\frac{d}{dt}\vec{\mathbf{x}} = A\vec{\mathbf{x}}$$

where A has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 1$ corresponding to eigenvectors $\vec{\xi}_1 = \binom{1}{1}$ and $\vec{\xi}_2 = \binom{1}{-1}$ respectively. Plot the **phase portrait** (vector field) of this system on the axes on the left below, and draw an **integral curve** of the system on it. On the axes on the right, draw the corresponding function $x_1(t)$.





6. **(12 points)** Transform the following second-order linear initial value problem

$$y'' + 7y' + 9y = t^3, \qquad y(\overset{0}{\cancel{1}}) = 1, \qquad y'(\overset{0}{\cancel{1}}) = 2,$$

into a first-order system of equations of the form

$$\frac{d}{dt}\vec{\mathbf{x}} = A\vec{\mathbf{x}} + \vec{\mathbf{f}}(t), \qquad \vec{\mathbf{x}}(0) = \vec{\mathbf{x}}_0.$$

Explicitly identify the matrix A and the vectors $\vec{\mathbf{f}}(t)$ and $\vec{\mathbf{x}}_0$. **You are not asked to solve!**

Set $x_1 = y$, $x_2 = y'$.

Then $x_1' = y' = x_2$, and $x_2' = y''$

Also, from the equation,

$$x_2' + 7x_2 + 9x_1 = t^3$$

Thus,

$$\begin{cases} x_1' = 0x_1 + x_2 + 0 \\ x_2' = -9x_1 - 7x_2 + t^3 \end{cases}$$

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -9 & -7 \end{pmatrix}}_{A}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ t^3 \end{pmatrix}}_{f(t)}$$

$$\vec{x}(0) = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} y(0) \\ y'(0) \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{\vec{x}_0}$$

7. **(12 points)** Find the Laplace transform of the function $f(t) = t$ from the definition of the Laplace transform. [The result is already in the table, but you are asked to derive it.]

$$\mathcal{L}[t] = \int_0^\infty t e^{-st} dt = -\frac{t}{s} e^{-st}\Big|_{t=0}^{t=\infty} - \int_0^\infty \left(-\frac{1}{s} e^{-st}\right) dt$$

↑ Integration by parts: $\begin{cases} u = t & dv = e^{-st} dt \\ du = dt & v = -\frac{1}{s} e^{-st} \end{cases}$

Notice: $\lim_{t\to\infty} -\frac{t}{s} e^{-st} = 0$ (L'Hôspital's rule).

Thus, $\mathcal{L}(t) = \frac{1}{s}\int_0^\infty e^{-st} dt = \frac{1}{s}\left(-\frac{1}{s} e^{-st}\right)\Big|_0^\infty = 0 - \frac{-1}{s^2} e^{-s\cdot 0} = \frac{1}{s^2}$

8. **(5 points)** Let $\delta_c(t) = \delta(t-c)$ be the Dirac-delta function centered at c and u_c be the Heaviside function with jump at c.

(a) Show that for $c > 0$, we have $\mathcal{L}[\delta_c] = s\mathcal{L}[u_c] - u_c(0)$. (Hint: compute both sides.)

$\mathcal{L}[\delta_c] = e^{-sc}$

$s\mathcal{L}[u_c] = s \cdot \frac{1}{s} e^{-sc} = e^{-sc}$

Also, $u_c(0) = 0$, since $c > 0$.

Thus, $\mathcal{L}[\delta_c] = s\mathcal{L}[u_c] - u_c(0)$.

(b) What relationship does this suggest between δ_c and u_c?

From the relationship $\mathcal{L}[y'] = s\mathcal{L}[y] - y(0)$, part (a) suggests that $u_c' = \delta_c$, i.e. Dirac-δ is the "derivative" of Heaviside in some sense.

9. **(12 points)** Find the general solution to the following system.

$$\frac{d}{dt}\vec{\mathbf{x}} = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix} \vec{\mathbf{x}}$$

Hint: The characteristic equation for the matrix is $\lambda^2 - 6\lambda + 9 = 0$, and $\begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$.

$\lambda^2 - 6\lambda + 9 = (\lambda - 3)^2$, so $\lambda = 3$ is a repeated eigenvalue.

Let $A = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}$. The hint shows $A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, so $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is the eigenvector for $\lambda = 3$. It remains to find the generalized eigenvector. Set $(A - 3I)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Then $\begin{pmatrix} 2-3 & 1 \\ -1 & 4-3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \begin{cases} -x + y = 1 \\ -x + y = 1 \end{cases}$ Choose $x = 0$. Then $y = 1$, $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Thus, general solution is $\boxed{\vec{x}(t) = c_1 e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2\left(te^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{3t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)}$

10. **(12 points)** Solve the following initial value problem.

$$y'' + 4y' + 13y = \delta_5(t), \qquad y(0) = 0, \qquad y'(0) = 0.$$

Apply $\mathcal{L}[\cdot]$:

$$(s^2 + 4s + 13)\,\mathcal{L}[y] = e^{-5t}$$

$$\Rightarrow \quad \mathcal{L}[y] = \frac{1}{s^2+4s+13}\, e^{-5t}$$

complete the square

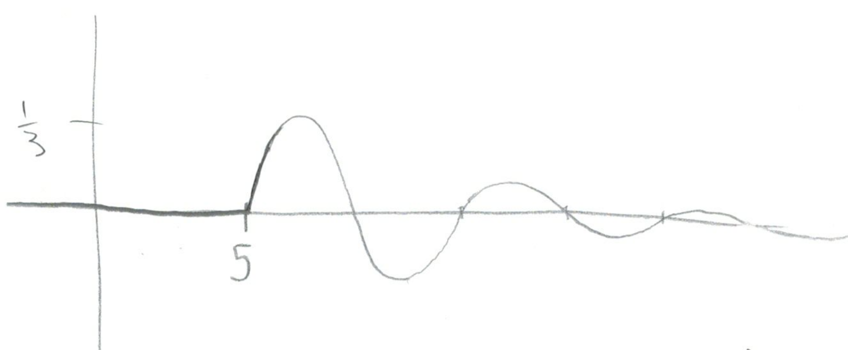
$$= \frac{1}{3}\,\frac{3}{(s+2)^2 + 3^2}\, e^{-5t}$$

$$\Rightarrow \quad y(t) = \frac{1}{3} e^{-2(t-5)} \sin(3(t-5))\, u_5(t)$$

Ok to stop here

$$= \begin{cases} 0 & , t < 5 \\ \frac{1}{3} e^{-2(t-5)} \sin(3(t-5)) & , t \geq 5 \end{cases}$$

rough sketch (not required):



not to scale