MATH 447/847
HOMEWORK 1 SOLUTIONS
SPRING 2015

To do Problem 2.1, it is helpful to have the following lemma (although problem 2.1 can be done directly).

Lemma 2.1. Suppose that a matrix A is upper-triangular and has an inverse B, then B is also upper-triangular.

Proof. Proof by induction. Consider a base case of a two by two matrix. Recall that $AB = I$. Therefore, $a_{21}b_{11} + a_{22}b_{21} = 0$. However, we know that $a_{21} = 0$ because the matrix is upper-triangular. Therefore, $a_{22}b_{21} = 0$. Note that a_{22} does not need to equal zero, therefore, b_{21} must be zero for the statement $AB = I$ to be generally true. Therefore, B must also be upper-triangular.

Consider an $n \times n$ matrix A and it's inverse B. Note that for any column j except $j = n$ that $0 = \sum_{i=1}^{n} a_{ni}b_{ij}$. However, a_{nn} is the only non-zero entry of A in this sum. Thus $0 = a_{nn}b_{nj}$. Since a_{nn} is not required to be zero, it must be that b_{nj} is always zero for $j \neq n$.

Since A and B are inverses, $AB = BA = I$. Therefore, considering $BA = I$ we can similarly multiply row j of B with column 1 of A. For all $j \neq 1$, we know that $0 = \sum_{i=1}^{n} a_{ni}b_{ij} = a_{11}b_{j1}$. Since a_{11} is not necessarily zero, it must be that $b_{j1} = 0$.

By induction, the inverse of A without the first column and row is upper-triangular. We have shown that the first column and last rows satisfy the conditions for upper triangular. Therefore, the inverse of an $n \times n$ upper-triangular matrix is upper-triangular. □

Corollary 2.1. Suppose that a matrix A is lower-triangular and has an inverse B, then B is also lower-triangular.

Proof. Since A is lower-triangular, A^* is upper-triangular, so by the previous lemma, $(A^{-1})^* = (A^*)^{-1}$ is upper triangular. Thus A^{-1} is lower-triangular. □

Problem 2.1. Show that if a matrix A is both triangular and unitary, then it is diagonal.

Proof. Without loss of generality, assume that A is lower-triangular. Since A is unitary, $A^* = A^{-1}$. Since all entries above the diagonal in A are 0, all entries below the diagonal are 0 in $A*$. Thus A^{-1} is clearly upper-triangular. By Lemma 2.1, A is also upper-triangular. Since A is both upper and lower triangular, it must be diagonal. □

Problem 2.3 (a). Let $A \in \mathbb{C}^{m \times m}$ be hermitian. An eigenvector of A is a **nonzero** vector $x \in \mathbb{C}^m$ such that $Ax = \lambda x$ for some $\lambda \in \mathbb{C}$, the corresponding eigenvalue. Prove that all eigenvalues of A are real.

Proof. Observe,

$$Ax = \lambda x$$
$$x^*Ax = x^*\lambda x$$
$$x^*Ax = \lambda x^*x.$$

Also,

$$(Ax)^* = (\lambda x)*$$
$$x^*A^* = x^*\overline{\lambda}$$
$$x^*A^*x = \overline{\lambda}x^*x.$$

Since A is self-adjoint,

$$\lambda x^*x = x^*(\lambda x) = x^*Ax = x^*A^*x = (Ax)^*x = (\lambda x)^*x = \overline{\lambda}x^*x.$$

Since $x \neq \vec{0}$, $x^*x = \|x\|^2 \neq 0$, so we can divide by x^*x to find that $\lambda = \overline{\lambda}$. This can only be true if λ is real. Therefore, all eigenvalues of a self-adjoint matrix are real.

□

Problem 2.3 (b). Prove that if x and y are eigenvectors corresponding to distinct eigenvalues, then x and y are orthogonal.

Proof. Suppose that λ_1, λ_2 are the eigenvalues corresponding to eigenvectors x and y respectively and $\lambda_1 \neq \lambda_2$.

First note that $x \cdot y = x^*y$. Observe,

$$\begin{aligned}\lambda_1 x^*y &= (\lambda_1 x)^*y \\ &= (Ax)^*y \\ &= x^*Ay \\ &= x^*(Ay) \\ &= x^*\lambda_2 y\end{aligned}$$

Therefore, $(\lambda_1 - \lambda_2)(x^*y) = 0$. Since $\lambda_1 \neq \lambda_2$ it follows that $(x^*y) = x \cdot y = 0$. In conclusion, x and y must be orthogonal. □

Problem 2.5(a). Let $S \in \mathbb{C}^{m \times m}$ be skew-hermitian ($S^* = -S$). Show by using exercise 2.3 that the eigenvalues of S are pure imaginary.

Proof. Let $\lambda \in \mathbb{C}$ be an eigenvalue of S. Observe,

$$Sx = \lambda x$$
$$x^*Sx = x^*\lambda x$$
$$x^*Sx = \lambda x^*x$$

Also,

$$(Sx)^* = (\lambda x)*$$
$$x^*S^* = x^*\overline{\lambda}$$
$$x^*S^*x = \overline{\lambda}x^*x$$

Since A is skew-hermitian, $\lambda x^*x = x^*Ax = -1(x^*A^*x) = -1(\overline{\lambda}x^*x)$. It follows that $\lambda = -\overline{\lambda}$. This can only be true if λ is imaginary. Therefore, all eigenvalues of of a skew-hermitian matrix are imaginary. □

Problem 2.5(b). Show that $I - S$ is nonsingular (invertible).

Proof. By the invertible matrix theorem, if $(I-S)$ is singular then there exists a solution to $(I-S)x = 0$ where x is not the zero vector.

Assume that $(I-S)$ is singular. Therefore, $(I-S)x = 0$ and $Ix = Sx$. Since x is non-zero and S invertible, there exists an eigenvalue λ such that $Sx = \lambda x$. However, the only possible solution when $Ix = Sx = \lambda x$ is that $\lambda = 1$. This contradicts part a which shows that all eigenvalues are imaginary. Therefore, we contradict our assumption that $(I-S)$ is singular and it must be nonsingular. □

Problem 2.5(c). Show that the matrix $Q = (I-S)^{-1}(I+S)$, known as the Cayley transform of S, is unitary. (This is a matrix analogue of a linear fractional transformation $(1+s)(1-s)$, which maps the left half of the complex s-plane conformally onto the unit disk.)

Proof. First consider Q^*. Note,

$$Q^* = ((I-S)^{-1}(I+S))^* = (I+S)^*(I-S)^{-*} = (I-S)(I+S)^{-1}.$$

Therefore, $QQ* = (I-S)^{-1}(I+S)(I-S)(I+S)^{-1}$. Note, $(I+S)(I-S) = (I+S)I - (I+S)S = I+S-S-SS$. Further $(I-S)(I+S) = (I-S)I + (I-S)S = I-S+S-SS$. It follows that $(I+S)(I-S) = (I-S)(I+S)$.

In conclusion, $QQ* = (I-S)^{-1}(I+S)(I-S)(I+S)^{-1} = (I-S)^{-1}(I-S)(I+S)(I+S)^{-1} = I$. Therefore, Q must be unitary. □