# MATH 447/847
# HOMEWORK 2 SOLUTIONS
# SPRING 2015

**Problem 3.1.** Prove that if $W$ is an arbitrary nonsingular matrix, the function $||\cdot||_W$ defined by (3.3) is a vector norm.

*Proof.* The function is a norm if $||x||_W = 0$ if and only if $x = \vec{0}$, $||cx||_W = |c|||x||_W$ for a scalar $c$, and it satisfies the triangle inequality.

Note $||x|| \geq 0$ for all $x \neq \vec{0}$. Further, $W$ cannot be the zero matrix because it is nonsingular. Thus, $||x||_w = ||Wx||$ and $Wx = \vec{0}$ if and only if $x$ is the zero vector. Therefore, $||x||_W = 0$ if and only if $x = \vec{0}$.

Let $c$ be a scalar. Observe, $||cx||_W = ||cWx|| = |c|||Wx|| = |c|||x||_W$. Therefore, $||cx||_W = |c|||x||_W$.

Let $x, y$ be vectors. Observe, $||x+y||_W = ||W(x+y)|| = ||Wx+Wy|| \leq ||Wx|| + ||Wy|| = ||x||_W + ||y||_W$. Therefore, $||x+y||_W \leq ||x||_W + ||y||_W$.

Since the function satisfies all three criteria, it is a norm. □

**Problem 3.2.** Let $||\cdot||$ denote any norm on $\mathbb{C}^m$ and also the induced matrix norm on $\mathbb{C}^{m\times m}$. Show that $\rho(A) \leq ||A||$, where $\rho(A)$ is the spectral radius of $A$, i.e., the largest absolute value $|\lambda|$ of an eigenvalue $\lambda$ of $A$.

*Proof.* Let $\lambda$ by any eigenvalue of $A$ with corresponding eigenvector to $x$. Note that $|\lambda|||x|| = ||\lambda x|| = ||Ax|| \leq ||A||||x||$. This implies that $|\lambda| \leq ||A||$ for all eigenvalues $\lambda$. Since $\rho(A)$ is defined to be the absolute value of the largest (in absolute value) eigenvalue, we have $\rho(A) \leq ||A||$. □

**Problem 3.3.** Vector and matrix $p$-norms are related by various inequalities, often involving the dimensions $m$ or $n$. For each of the following, verify the inequality and give an example of a non-zero vector or matrix (for general, $m, n$) for which the equality is achieved. In this problem $x$ is an $m$-vector and $A$ is an $m \times n$ matrix.

(1) $||x||_\infty \leq ||x||_2$
Observe, $||x||_\infty = \max_{1\leq i\leq m} |x_i| = \max_{1\leq i\leq m} \sqrt{|x_i|^2)} \leq ||x||_2$. Equality is achieved when the vector has a singular non-zero value.

(2) $||x||_2 \leq \sqrt{m}||x||_\infty$
Observe $||x||_2 = \sqrt{\sum_{i=1}^{m} |x_i|^2} \leq \sqrt{\sum_{i=1}^{m} |\max_{1\leq j\leq m} x_i|^2} = \sqrt{m}\max_{1\leq j\leq m} x_i = \sqrt{m}||x||_\infty$. Equality is attained when $x$ is the one vector.

(3) $||A||_\infty \leq \sqrt{n}||A||_2$
Observe,$||A||_\infty = \sup_{x\neq 0} \frac{||Ax||_\infty}{||x||_\infty}$. By part a, $\sup_{x\neq 0} \frac{||Ax||_\infty}{||x||_\infty} \leq \sup_{x\neq 0} \frac{||Ax||_2}{||x||_\infty}$. Rearranging part $b$, $\frac{1}{\sqrt{m}}||x||_2 \leq ||x||_\infty$. Therefore, $\sup_{x\neq 0} \frac{||Ax||_2}{||x||_\infty} \leq \sup_{x\neq 0} \frac{\sqrt{n}||Ax||_2}{||x||_2} = \sqrt{n}||x||_2$. In conclusion, $||A||_\infty \leq \sqrt{n}||A||_2$.

Equality is attained when $A$ is a matrix with one row of all ones and zeros elsewhere. Since $||A||_\infty$ is the max row sum, it is $n$. Now $||A||_2 = \sup_{||x||=1} \sum_{i=1}^{m} s_i$ where $s_i$ is the $i$th row sum of $A$ (because $||x|| = 1$). Thus $||A||_2 = \sqrt{n}$ and $\sqrt{n}||A||_2 = n$.

(4) $||A||_2 \leq \sqrt{m}||A||_\infty$

Observe, $||A||_2 = \sup_{x\neq 0} \frac{||Ax||_2}{||x||_2}$. By part b, $\sup_{x\neq 0} \frac{||Ax||_2}{||x||_2} \leq \sup_{x\neq 0} \frac{\sqrt{m}||Ax||_\infty}{||x||_2}$. By part a, $\sup_{x\neq 0} \frac{\sqrt{m}||Ax||_\infty}{||x||_2} \leq \sup_{x\neq 0} \frac{\sqrt{m}||Ax||_\infty}{||x||_\infty} = \sqrt{m}||A||_\infty$. Therefore, $||A||_2 \leq \sqrt{m}||A||_\infty$.

Equality is attained when $A$ is a matrix with one column of all ones and zeros elsewhere. Note $||A||_\infty = 1$ because it is the max row sum. Thus $\sqrt{m}||A||_\infty = \sqrt{m}$. Further $||A||_2 = \sqrt{\sum_{i=1}^{m} 1} = \sqrt{m}$. Thus, $||A||_2 = \sqrt{m}||A||_\infty = \sqrt{m}$.

**Problem 3.6(a).** Let $||\cdot||$ denote any norm on $\mathbb{C}^m$. The corresponding dual norm $||\cdot||'$ is defined by the formula $||x||' = sup_{||y||=1}|y * x|$. Prove that $||\cdot||'$ is a norm.

*Proof.* We must show that $\|\cdot\|'$ satisfies the three norm axioms.

Step 1: Show that $||x||' = 0$ if and only if $x = \vec{0}$.

Suppose that $||x||' = 0$. It follows that $sup_{||y||=1}|y^*x| = 0$. Since the maximum of the positive values $|y^*x|$ is zero, we must have $|y^*x| = 0$ (i.e., $y^*x = 0$) for all $y$ such that $\|y\| = 1$. We can choose a special $y$ to show that $x$ must the be zero. Suppose that $x \neq \vec{0}$. Then let us choose $y = x/\|x\|$. Then $\|y\| = 1$, and

$$\|x\| = \frac{\|x\|^2}{\|x\|} = \frac{x^*x}{\|x\|} = \frac{x^*}{\|x\|}x = \left(\frac{x}{\|x\|}\right)^* x = y^*x = 0$$

So our assumption that $x \neq \vec{0}$ is false, and we must have $x = \vec{0}$. Thus, if $||x||' = 0$ then $x = \vec{0}$.

Next, suppose $x = \vec{0}$. Then $|y^*\vec{0}| = 0$ for any $y$. Therefore, if $x = \vec{0}$ then $||x||' = \sup_{||y||=1} |y^*x| = \sup_{||y||=1} 0 = 0$.

Step 2: Show that $||ax||' = |a|||x||'$ for any scalar $a$ and vector $x$.

Let $a$ be a scalar and $x$ a vector. Observe $||ax||' = \sup_{||y||=1} |y^*ax| = \sup_{||y||=1} |a||y^*x| = |a| \sup_{||y||=1} |y^*x| = |a|||x||'$. Therefore, $||ax||' = |a|||x||'$.

Step 3: Show that $||x+z||' \leq ||x||' + ||z||'$ for all vectors $x$ and $z$.

Let $x, z$ be vectors. Observe, $||x+z||' = \sup_{||y||=1} |y^*(x+z)| = \sup_{||y||=1} |y^*x + y^*z| \leq \sup_{||y||=1} |y^*x| + |y^*z| = \sup_{||y||=1} |y^*x + \sup_{||y||=1} |y^*z| = ||x||'||z||'$. Therefore, $||x+z||' \leq ||x||' + ||z||'$.

Since $||\cdot||'$ satisfies all three conditions, it is a norm. □