MATH 447/847
HOMEWORK 3 SOLUTIONS
SPRING 2015

Problem 4.1(a). Determine the SVD of $A = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$

Note, $A^*A = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}$. Suppose $x \neq \vec{0}$ and $(A^*A - \lambda I)x = 0$. It must be that $\det(A^*A - \lambda I) = \det \begin{bmatrix} 9-\lambda & 0 \\ 0 & 4-\lambda \end{bmatrix} = 0$. Thus, $(9-\lambda)(4-\lambda) = 0$ and $\lambda_1 = 9$, $\lambda_2 = 4$.

For the first eigenvector consider $(A^*A - \lambda_1 I)x = 0$. Observe, $(A^*A - \lambda_1 I)x = \begin{bmatrix} 0 & 0 \\ 0 & -5 \end{bmatrix} x = 0$. Thus, $-5x_2 = 0$, $x_2 = 0$ and x_1 is free. Let $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

For the second eigenvector consider $(A^*A - \lambda_2 I)x = 0$. Observe, $(A^*A - \lambda_2 I)x = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix} x = 0$. Thus, $5x_1 = 0$, $x_1 = 0$ and x_2 is free. Let $x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

It follows that $V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Observe, $\Sigma = \begin{bmatrix} \sqrt{9} & 0 \\ 0 & \sqrt{4} \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$. Recall U is AV with each element of column i divided by σ_i. Thus, $AV = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

In conclusion, $A = U\Sigma V^*$ as they are defined above and thus they compose the SVD of A. That is,

$$\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \sqrt{9} & 0 \\ 0 & \sqrt{4} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^*$$

Problem 4.1(b). Determine the SVD of $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

Note, $A^*A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}$. Suppose $x \neq \vec{0}$ and $(A^*A - \lambda I)x = 0$. It must be that $\det(A^*A - \lambda I) = \det \begin{bmatrix} 4-\lambda & 0 \\ 0 & 9-\lambda \end{bmatrix} = 0$. Thus, $(9-\lambda)(4-\lambda) = 0$ and $\lambda_1 = 9$, $\lambda_2 = 4$.

For the first eigenvector consider $(A^*A - \lambda_1 I)x = 0$. Observe, $(A^*A - \lambda_1 I)x = \begin{bmatrix} -5 & 0 \\ 0 & 0 \end{bmatrix} x = 0$. Thus, $-5x_1 = 0$, $x_1 = 0$ and x_2 is free. Let $x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

For the second eigenvector consider $(A^*A-\lambda_2 I)x=0$. Observe, $(A^*A-\lambda_2 I)x=\begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}x=0$. Thus, $5x_2=0$, $x_2=0$ and x_1 is free. Let $x=\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

It follows that $V=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Observe, $\Sigma=\begin{bmatrix} \sqrt{9} & 0 \\ 0 & \sqrt{4} \end{bmatrix}=\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$. Note, $AV=\begin{bmatrix} 0 & 3 \\ 2 & 0 \end{bmatrix}$. Thus, $U=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

In conclusion, $A=U\Sigma V^*$ as they are defined above and thus they compose the SVD of A.

Problem 4.1(c). Determine the SVD of $A=\begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

Note, $A^*A=\begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}$. Suppose $x\neq\vec{0}$ and $(A^*A-\lambda I)x=0$. It must be that $\det(A^*A-\lambda I)=\det\begin{bmatrix} 4-\lambda & 0 \\ 0 & -\lambda \end{bmatrix}=0$. Thus, $(4-\lambda)(-\lambda)=0$ and $\lambda_1=4$, $\lambda_2=0$.

For the first eigenvector consider $(A^*A-\lambda_1 I)x=0$. Observe, $(A^*A-\lambda_1 I)x=\begin{bmatrix} 0 & 0 \\ 0 & -4 \end{bmatrix}x=0$. Thus, $-4x_2=0$, $x_2=0$ and x_1 is free. Let $x=\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

For the second eigenvector consider $(A^*A-\lambda_2 I)x=0$. Observe, $(A^*A-\lambda_2 I)x=\begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}x=0$. Thus, $4x_1=0$, $x_1=0$ and x_2 is free. Let $x=\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

It follows that $V=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Note that $A^*A=\begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Suppose $x\neq\vec{0}$ and $(AA^*-\lambda I)x=0$. It must be that $\det(AA^*-\lambda I)=\det\begin{bmatrix} 4-\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix}=0$. Thus, $(4-\lambda)(\lambda^2)=0$ and $\lambda_1=4$, $\lambda_2=0$, $\lambda_3=0$.

For the first eigenvector consider $(AA^*-\lambda_1 I)x=0$. Observe, $(AA^*-\lambda_1 I)x=\begin{bmatrix} 0 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix}x=0$. Thus, $-4x_2=0$, $-4x_3=0$ so $x_2=x_3=0$ and x_1 is free. Let $x=\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

For the second and third eigenvectors $(AA^*-\lambda_2 I)x=(AA^*-\lambda_3 I)x=0$. In each case $4x_1=0$ and x_1,x_2 are free. Let the respective eigenvectors for λ_2,λ_3 be $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

It follows that $U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Observe, $\Sigma = \begin{bmatrix} \sqrt{4} & 0 \\ 0 & \sqrt{0} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$.

In conclusion, $A = U\Sigma V^*$ as they are defined above and thus they compose the SVD of A.

Problem 2. Determine the SVD of $A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$

Note, $A^*A = \begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{bmatrix}$. Suppose $x \neq \vec{0}$ and $(A^*A - \lambda I)x = 0$. It must be that $\det(A^*A - \lambda I) = \det \begin{bmatrix} 13-\lambda & 12 & 2 \\ 12 & 13-\lambda & -2 \\ 2 & -2 & 8-\lambda \end{bmatrix} = 0$. Thus, $(25-\lambda)(\lambda-9)(-\lambda) = 0$ and $\lambda_1 = 25$, $\lambda_2 = 9$, and $\lambda_3 = 0$.

Clearly, $\Sigma = \begin{bmatrix} 5 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix}$.

To construct V consider the eigenvectors of A^*A. For the first eigenvector consider $(A^*A - \lambda_1 I)x = 0$. Observe, $(A^*A - \lambda_1 I)x = \begin{bmatrix} -12 & 12 & 2 \\ 12 & -12 & -2 \\ 2 & -2 & -17 \end{bmatrix} x = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x = 0$. Thus, $x_1 = x_2$ and $x_3 = 0$. An eigenvector of unit length is $x = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$.

For the second eigenvector consider $(A^*A - \lambda_2 I)x = 0$. Observe, $(A^*A - \lambda_2 I)x = \begin{bmatrix} 4 & 12 & 2 \\ 12 & 4 & -2 \\ 2 & -2 & -1 \end{bmatrix} x = \begin{bmatrix} 1 & 0 & -1/4 \\ 0 & 1 & 1/4 \\ 0 & 0 & 0 \end{bmatrix} = 0$. Thus, $x_1 = 4x_3$, $x_2 = -4x_3$, and x_3 is free. An eigenvector perpendicular to the first is $x = \begin{bmatrix} \frac{-1}{3\sqrt{2}} \\ \frac{1}{3\sqrt{2}} \\ \frac{-2\sqrt{2}}{3} \end{bmatrix}$.

For the third eigenvector consider $(A^*A-\lambda_3 I)x=0$. Observe, $(A^*A-\lambda_3 I)x=\begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{bmatrix} x = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} = 0$. Thus, $x_1 = -2x_3$, $x_2 = 2x_3$, and x_3 is free. A unit eigenvector perpendicular to the first two is $x = \begin{bmatrix} \frac{-2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$.

It follows that $V = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{3\sqrt{2}} & \frac{-2}{3} \\ \frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} & \frac{2}{3} \\ 0 & \frac{-2\sqrt{2}}{3} & \frac{1}{3} \end{bmatrix}$.

Observe $AV = \begin{bmatrix} \frac{5}{\sqrt{2}} & \frac{-3}{\sqrt{2}} \\ \frac{5}{\sqrt{2}} & \frac{3}{\sqrt{2}} \end{bmatrix}$. It follows that $U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$.

In conclusion, $A = U\Sigma V^*$ as they are defined above and thus they compose the SVD of A.