MATH 447/847
HOMEWORK 4 SOLUTIONS
SPRING 2015

Problem 20.2. Suppose $A \in \mathbb{C}^{m\times m}$ satisfies the condition of exercise 20.1 (principle minors are all nonsingular) and is banded with bandwidth $2p+1$, i.e., $a_{ij} = 0$ for $|i-j| > p$. What can you say about he sparsity patterns of the factors L and U of A.

Proof. By the condition of exercise 20.1, A has an LU-Factorization $A = LU$. Since $A = LU$, $a_{ij} = \sum_{k=1}^{m} l_{ik}u_{kj}$.

Consider when $i-j > p$ and thus $a_{ij} = 0$. Since U is upper triangular $u_{kj} = 0$ for $k > j$. So $a_{ij} = 0 = \sum_{k=1}^{j} l_{ik}u_{kj}$ for $i-j > p$. Note that when $j = 1$, $0 = \sum_{k=1}^{1} l_{ik}u_{kj} = l_{i1}u_{11}$. Since u_{11} need not be zero, $l_{i1} = 0$. By induction assume that all $l_{ik} = 0$ for a given i and $k < j$. It follows that $0 = 0 + \cdots + 0 + l_{ij}u_{jj}$ and thus $l_{ij} = 0$. Therefore $l_{ij} = 0$ for all $i-j > p$. Since L is lower triangular, it also satisfies $l_{ij} = 0$ for all $j-i > p$. Thus $l_{ij} = 0$ for all $|i-j| > p$.

Similarly consider when $j-i > p$ and thus $a_{ij} = 0$. Since L is lower triangular $l_{ik} = 0$ for $k > i$. So $a_{ij} = 0 = \sum_{k=1}^{i} l_{ik}u_{kj}$ for $j-i > p$. Note that when $i = 1$, $0 = \sum_{k=1}^{1} l_{ik}u_{kj} = l_{11}u_{1j}$. Since l_{11} need not be zero, $u_{1j} = 0$. By induction, $u_{ij} = 0$. Therefore, $u_{ij} = 0$ when $j-i > p$. Since U is upper triangular, it also satisfies $u_{ij} = 0$ for all $i-j > p$. Thus $u_{ij} = 0$ for all $|i-j| > p$.

In conclusion, L satisfies $l_{ij} = 0$ when $|i-j| > p$ and U satisfies $u_{ij} = 0$ when $|i-j| > p$. □

Problem 20.4. Rewrite Algorithm 20.1 to have one explicit for loop indexed by k. Inside this loop, U will be updated at each step by a certain rank-one outer product. This "outer product" form of Gaussian elimination may be a better starting point than algorithm 20.1 if one wants to optimize computer performance.

for $k = 1$ **to** $m-1$ **do**
  $l_{k+1:m,k} = \frac{u_{k+1:m,k:m}}{u_{kk}}$
  $u_{k+1:m,k:m} = u_{k+1:m,k:m} - l_{k+1:m,k:m}u_{k,k:m}$
end for

Problem 21.1. Let A be the 4 by 4 matrix (20.3) considered in this lecture and the previous one.

(a) Determine $\det A$ from (20.5).
Recall, that 20.5 gives the $A = LU$ decomposition

$$\begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 4 & 3 & 1 & 0 \\ 3 & 4 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

By determinant rules $\det A = \det L \det U$. Thus $\det A = 1(1(1-0)) \cdot 2(2(2-0)) = 8$.

(b) Determine $\det A$ from (21.3). Recall, that 21.3 gives the $PA = LU$ decomposition

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3/4 & 1 & 0 & 0 \\ 1/2 & -2/7 & 1 & 0 \\ 1/4 & -3/7 & 1/3 & 1 \end{bmatrix} \begin{bmatrix} 8 & 7 & 9 & 5 \\ 0 & 7/4 & 9/4 & 17/4 \\ 0 & 0 & -6/7 & -2/7 \\ 0 & 0 & 0 & 2/3 \end{bmatrix}.$$

Notice that $\det P = -1$. Thus by the determinate rules $\det PA = -\det A = \det L \det U$. Therefore, $\det A = (-1) \cdot 1(1(1-0 \cdot \frac{1}{3})) \cdot \frac{2}{3}(\frac{-6}{7}(\frac{7}{4}8 - 0 \cdot 7)) = 8$.

Problem 21.6. Suppose $A \in \mathbb{C}^{m \times m}$ is strictly column diagonally dominant, which means that for each k, $|a_{kk}| > \sum_{j \neq k} |a_{jk}|$. Show that if Gaussian elimination with partial pivoting is applied to A, no row interchanges take place.

Proof. Proof by induction on the number of columns. Base case, consider a column diagonally dominant matrix $A \in \mathbb{C}^{m \times 1}$. Since A is strictly column diagonally dominant, it follows that a_{11} is the largest element in the first column. Recall that partial pivoting only exchanges rows and pivots on the largest element in a column. Thus, a_{11} is the desired pivot and no rows are exchanged.

Induction step, consider a column diagonally dominant matrix $A \in \mathbb{C}^{m \times n}$. As in the base case, diagonal dominance ensures that no row exchanges are made in selecting the first pivot.

Let A' denote the matrix after pivoting in column 1. Notice that for every column k,

$$\begin{aligned} |a'_{kk}| &= |a_{kk} - \frac{a_{k1}}{a_{11}} a_{1k} \\ &\geq |a_{kk}| - |\frac{a_{k1}}{a_{11}} a_{1k}| \\ &> \sum_{i \neq k} \left(|a_{ik}| - |\frac{a_{k1}}{a_{11}} a_{1k}| \right) && \text{(By diagonal dominance of A)} \\ &= \sum_{i \neq k} |a'_{ik}| \end{aligned}$$

Therefore, $|a'_{kk}| > \sum_{i \neq k} |a'_{ik}|$ and A' is diagonally dominant and no row exchanges where made. Since the first column is now zeros except for the first row, row operations of subsequent pivots will not affect the first column. Therefore, A' without the first column is a diagonally dominant matrix with fewer columns. By induction, no row exchanges are required for Gaussian elimination on the diagonally dominant matrix A. □