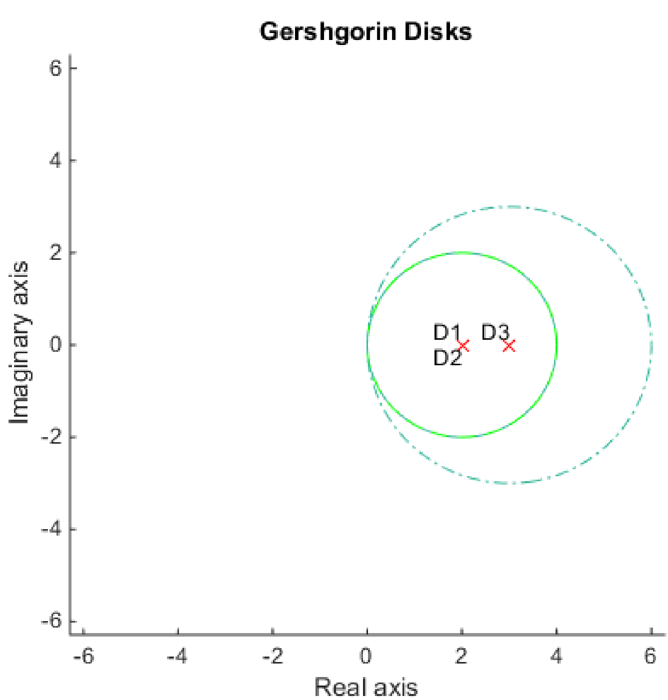
MATH 447/847
HOMEWORK 5 SOLUTIONS
SPRING 2015

Problem 1. Given a square matrix $A = (a_{ij})_{i,j=1}^{n}$, let us define its **Gerschgorin disks** for $i = 1,\ldots,n$ by:

$$D_i = \left\{ z \in \mathbb{C} : |z - a_{ii}| \le \sum_{\substack{j=1 \\ j \ne i}}^{n} |a_{ij}| \right\}$$

These are disks in the complex plane $\mathbb{C}$, which are centered at the diagonal entries a_{ii}, and whose radius is the sum of the absolute values of the off-diagonal entries in the i^{th} row. They are very important tools in numerical analysis. Draw a picture (on the same plane) of the Gerschgorin disks D_1, D_2, D_3 for the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 3. \end{bmatrix}$$



Problem 24.2 (a). Here is Gerschgorin's theorem, which holds for any $m \times m$ matrix A, symmetric or nonsymmetric. Every eigenvalue of A lies in at least one of the m circular disks in the complex plane with centers a_{ii} and radii $\sum_{j \neq i} |a_{ij}|$. Moreover, if n of these disks form a connected domain that is disjoint from the other $m-n$ disks, then there are precisely n eigenvalues of A within this domain. Prove the first part of Gerschgorin's theorem. (Hint: Let λ be any eigenvalue of A, and the x corresponding eigenvector with largest entry 1.)

Proof. Let λ be any eigenvalue of A, and the x corresponding eigenvector with largest entry $x_i = 1$ in row i. This is guaranteed because for any eigenvector, we can always scale it by the reciprocal of the largest magnitude entry.

Since x is an eigenvector, $Ax = \lambda x$. For each row k, $\sum_j a_{kj} x_j = \lambda x_k$. It follows $\sum_{j \neq k} a_{kj} x_j = \lambda x_k - a_{kk} x_k$. Specifically, $\sum_{j \neq i} a_{ij} x_j = \lambda x_i - a_{ii} x_i = \lambda - a_{ii}$.

Therefore $|\lambda - a_{kk}| \leq \sum_{j \neq k} a_{kj} x_j$ and satisfies the condition to be in the Gerschgorin disk. □

Problem 24.2 (c). Give estimates based on Gerschgorin's theorem for the eigenvalues of $A = \begin{bmatrix} 8 & 1 & 0 \\ 1 & 4 & \varepsilon \\ 0 & \varepsilon & 1 \end{bmatrix}$, $|\varepsilon| < 1$.

Solution. Since the matrix is symmetric, all eigenvalues are real. Further, the eigenvalues fall within the disks, therefore there are eigenvalues in the ranges $8 \pm 1, 4 \pm (1+\varepsilon), 1 \pm \varepsilon$. Since $|\varepsilon| < 1$, this is at worst $8 \pm 1, 4 \pm (2), 1 \pm 1$.

Problem 2(b). Gershgorin's Theorem (sometimes called Gershgorin's Localization Theorem) says that all the eigenvalues of a matrix must live in the Gershgorin disks. Without computing the eigenvalues of the matrix (it's not worth your time), tell whether the matrix in Problem 1 could have an eigenvalue greater than 6, or less than zero, based on your picture.

Solution. The matrix of problem 1 could not have an eigenvalue greater than 6 or less than zero because none of the circles radii cover those regions.

Problem 3. Consider trying to solve the problem $A\vec{x} = \mathbf{b}$ by an iteration method $\vec{x}^{k+1} = G\vec{x}^k + b$ for some G. If we write $A = D - L - U$, where D is a diagonal matrix made from the diagonal entries of A, $-L$ are the lower entries of A, and $-U$ are the upper entries of A then the Jacobi method is to choose $G = D^{-1}(L+U)$. That is, given $A = (a_{ij})_{i,j=1}^n$, set G to be

$$G = \begin{bmatrix} 0 & -\frac{a_{12}}{a_{11}} & -\frac{a_{13}}{a_{11}} & \cdots & -\frac{a_{1,n}}{a_{11}} \\ -\frac{a_{21}}{a_{22}} & 0 & -\frac{a_{23}}{a_{22}} & \cdots & -\frac{a_{2,n}}{a_{22}} \\ -\frac{a_{31}}{a_{33}} & -\frac{a_{32}}{a_{33}} & 0 & \cdots & -\frac{a_{3,n}}{a_{33}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{a_{n1}}{a_{nn}} & -\frac{a_{n2}}{a_{nn}} & -\frac{a_{n3}}{a_{nn}} & \cdots & 0 \end{bmatrix}.$$

Recall that a strictly-diagonally-dominant (SDD) matrix $A = (a_{ij})_{i,j=1}^n$, is a matrix such that

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \text{ for all } j = 1, \ldots, n.$$

Prove that if A is SDD, then the Jacobi method converges. [Hint: Use the Fundamental Theorem of Iterative Methods and Gershgorin's Theorem, which together make the proof just a couple of lines.]

Proof. By Gerschgorin, $|\lambda| \leq \sum_{j\neq i} |G_{ij}| = \frac{\sum_{j\neq i} |a_{ij}|}{a_{ii}}$. Because A is strictly diagonally dominant, $\sum_{j\neq i} |a_{ij}| < a_{ii}$ and $\left| \frac{\sum_{j\neq i} |a_{ij}|}{a_{ii}} \right| < 1$.

Therefore, $|\lambda| < 1$ for all eigenvalues λ of G. It follows that, $\rho(G) < 1$. By the Fundamental Theorem of Iterative Methods, the method converges. □

Lemma 4(a). The eigenvalues of A and $I - \tau A$ are bijective.

Proof. Suppose that $Ax = \lambda x$ for $x \neq \vec{0}$. Observe,

$$(I - \tau A)x = x - \tau Ax = x - \tau\lambda x = (1 - \tau\lambda)x.$$

Therefore, for every eigenvalue of A there is a corresponding eigenvalue $(1 - \tau\lambda)$ of $I - \tau A$.

Suppose that $(I - \lambda A)x = \mu x$ for $x \neq \vec{0}$. Observe

$$\begin{aligned} x - \tau Ax &= \mu x \\ x - \mu x &= \tau Ax \\ Ax &= \frac{1}{\tau}(1 - \mu)x. \end{aligned}$$

Let $\lambda = \frac{1}{\tau}(1 - \mu)$. Thus, for every eigenvalue μ, λ is a corresponding eigenvalue of A. Further, the map is linear and so the ordering of sets is the same. □

Problem 4(a). Consider Richard's iteration method with the scaling parameter $\tau > 0$ ($A^{-1} \approx B = \tau I$, $Q = B^{-1} = \frac{1}{\tau}I$), so that the iteration $\vec{x}^{k+1} = (I - BA)\vec{x}^k + Bb$ is just

$$\vec{x}^{k+1} = (I - \tau A)\vec{x}^k + \tau b,$$

so that $G = (I - \tau A)$. Recall the error equation $\vec{e}^{k+1} = G\vec{e}^k$. Show that

$$\|\vec{e}^{k+1}\|_2 \leq \left(\max_i |1 - \tau\lambda_i| \right) \|\vec{e}^k\|_2$$

where λ_i are the eigenvalues of A. Assume A is adjoint.

Proof. By definition, $\vec{e}^{k+1} = (I - \tau A)\vec{e}^k$. It follows that $||\vec{e}^{k+1}|| = ||(I - \tau A)\vec{e}^k||$. Recall, $||(I - \tau A)\vec{e}^k|| \leq ||(I - \tau A)||||\vec{e}^k||$. Since A is adjoint, $||A||_2 = \rho(A) = \max_i |\lambda_i|$. By Lemma 4(a), $\rho(I - \tau A) = \max_i |1 - \tau\lambda_i|$. It follows that $||(I - \tau A)|| = \max_i |(1 - \tau\lambda_i)|$.

In conclusion, $||\vec{e}^{k+1}|| \leq \max_i |(1 - \tau\lambda_i)|||\vec{e}^k||$. □

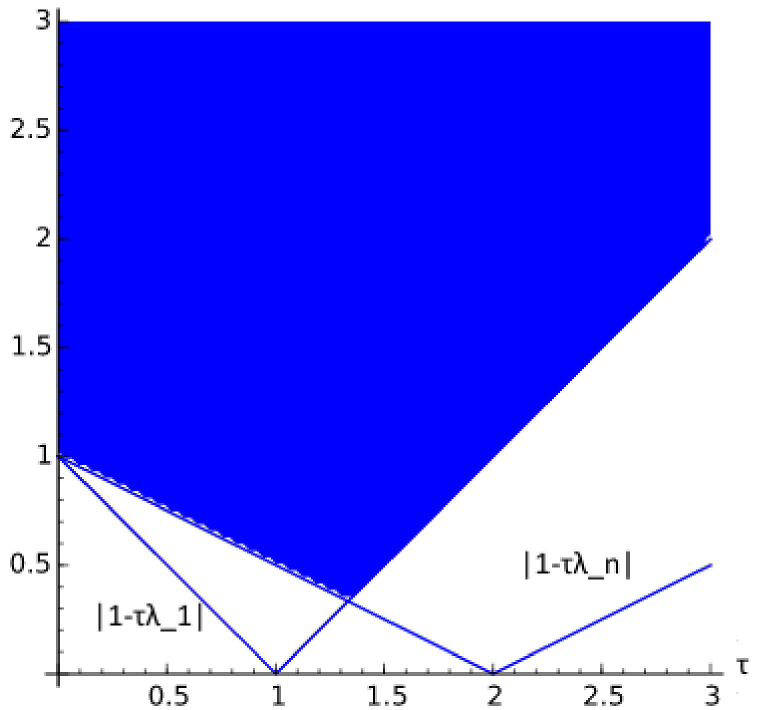
Problem 4(b). Let us order the eigenvalues so that $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Show that

$$\max_i |1 - \tau\lambda_i| = \max\{|1 - \tau\lambda_1|, |1 - \tau\lambda_n|\}$$

and that this quantity is **smallest** when $1 - \tau\lambda_1 = -1 + \tau\lambda_n$. (Hint: Don't use calculus to find the min of the max, it is ugly. Instead, just draw pictures of $|1 - \tau\lambda_i|$.) Also, solve for τ in this case.

Proof. Given ordered eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$, it is clear that $\max_i |1-\tau\lambda_i| = \max\{|1-\tau\lambda_1|, |1-\tau\lambda_n|\}$ because the expression gets small as $\tau\lambda_i$ approaches 1. Our ordering guarantees that either $\tau\lambda_1$ or $\tau\lambda_n$ is the furthest value from 1.

When we plot the functions $|1-\tau\lambda_1|, |1-\tau\lambda_n|$ versus τ. We know that $\max_i |1-\tau\lambda_i|$ is at least the value of the greater of these functions. Thus,



While the particular values of λ_1, λ_n can vary, the plots maintain this general relationship because $\lambda_1 \leq \lambda_n$. As can be seen in the graph, the minimum value of $\max_i |1-\tau\lambda_i|$ is where the $1-\tau\lambda_1 = -1+\tau\lambda_n$. At this point, $\tau = \frac{2}{\lambda_1+\lambda_n}$ $\square$