MATH 447/847
HOMEWORK 6 SOLUTIONS
SPRING 2015

Problem 1. Let $i = \sqrt{-1}$. Show that, for any integers m and n,

(a) $\int_0^{2\pi} e^{inx} e^{-imx}\, dx = \begin{cases} 2\pi & \text{if } m = n, \\ 0 & \text{if } m \neq n, \end{cases}$

Proof. Suppose $m = n$. Observe,

$$\int_0^{2\pi} e^{inx} e^{-imx}\, dx = \int_0^{2\pi} e^{(n-m)ix}\, dx = \int_0^{2\pi} e^0 = \int_0^{2\pi} 1 = 2\pi.$$

Suppose $m \neq n$. Observe,

$$\int_0^{2\pi} e^{inx} e^{-imx}\, dx = \int_0^{2\pi} e^{(n-m)ix}\, dx = \frac{1}{(n-m)i}[(e^{i2\pi})^{(n-m)} - e^{(n-m)i0}] = \frac{1}{(n-m)i}[1-1] = 0.$$

□

(b) $\int_0^{2\pi} \cos(nx)\cos(mx)\, dx = \begin{cases} \pi & \text{if } m = n \neq 0, \\ 0 & \text{if } m \neq n, \end{cases}$

Proof. Suppose $m = n$. Observe,

$$\begin{aligned} \int_0^{2\pi} \cos(nx)\cos(mx)\, dx &= \int_0^{2\pi} \cos^2(nx)\, dx = \int_0^{2\pi} \left(\frac{1}{2} + \frac{1}{2}\cos(2nx)\right) dx \\ &= \int_0^{2\pi} \frac{1}{2}\, dx + \frac{1}{2}\underbrace{\int_0^{2\pi} \cos(2nx)\, dx}_{\to 0} \\ &= \frac{1}{2} 2\pi = \pi \end{aligned}$$

The second integral is zero, since it is cosine (at a higher integer-frequency) integrated over its period. Next, suppose $m \neq n$. Using a standard trig identity for the product of cosines, we observe

$$\begin{aligned} \int_0^{2\pi} \cos(nx)\cos(mx)\, dx &= \int_0^{2\pi} \frac{1}{2}[\cos((n-m)x) + \cos((n+m)x)]\, dx \\ &= \frac{1}{2}\int_0^{2\pi} \cos((n-m)x)\, dx + \frac{1}{2}\int_0^{2\pi} \cos((n+m)x)\, dx \\ &= 0 + 0 = 0. \end{aligned}$$

The integrals are zero, since again we are integrating cosine (at a higher integer-frequency) over its period. □

(c) $\int_0^{2\pi} \sin(nx)\cos(mx)\, dx = 0$

Proof. Using a standard trig identity for the product of sine and cosine, we observe

$$\begin{aligned}\int_0^{2\pi} \sin(nx)\cos(mx)\,dx &= \int_0^{2\pi} \frac{1}{2}[\sin((n+m)x) + \sin((n-m)x)]\,dx \\ &= \frac{1}{2}\int_0^{2\pi} \sin((n+m)x)\,dx + \frac{1}{2}\int_0^{2\pi} \sin((n-m)x)\,dx \\ &= 0 + 0 = 0.\end{aligned}$$

The integrals are zero, since again we are integrating sine (at a higher integer-frequency) over its period. Note that when $n = m$, then $n - m = 0$, so $\sin((n-m)x) = \sin(0) = 0$, so this case also gives zero. ☐

(d) $\int_0^{2\pi} \sin(nx)\sin(mx)\,dx = \begin{cases} \pi & \text{if } m = n \neq 0, \\ 0 & \text{if } m \neq n \end{cases}$

Proof. We could do these in a similar way to the others, but here is another way. Note that, since $i^2 = -1$,

$$\begin{aligned}e^{inx}e^{-imx} &= (\cos(nx) + i\sin(nx))(\cos(mx) + i\sin(mx)) \\ &= \cos(nx)\cos(mx) - \sin(nx)\sin(mx) + i[\sin(nx)\cos(mx) + \cos(nx)\sin(mx)]\end{aligned}$$

Integrating both sides and using the above results gives the desired identity. This is the time saver mentioned below. ☐

Time saver: Do the first two (use a trig identity on the second one), then expand the exponents in the first one with Euler's formula, ($e^{i\theta} = \cos(\theta) + i\sin(\theta)$), to get the rest.

Problem 2. Consider the "Continuous to Discrete" Fourier transform, given by the relations

$$f(x) = a_0 + \sum_{k=1}^{\infty} (a_k \cos(kx) + b_k \sin(kx)) \tag{1}$$

(a) Find the Fourier coefficients a_0, a_k, and b_k in the case where $f(x) = x$. (Hint: Use the relationships in Problem 1 on (1) to isolate the coefficents. Then integrate by parts.) The coefficients will be numbers that depend only on k.

Solution. The formulas used were derived in class. Observe,

$$b_k = \frac{1}{\pi}\int_0^{2\pi} x\sin(kx)dx = \frac{1}{\pi}\left[\frac{\sin(kx) - kx\cos(kx)}{k^2}\right]_0^{2\pi} = \frac{0 - k2\pi + 0}{\pi k^2} = \frac{-2}{k}$$

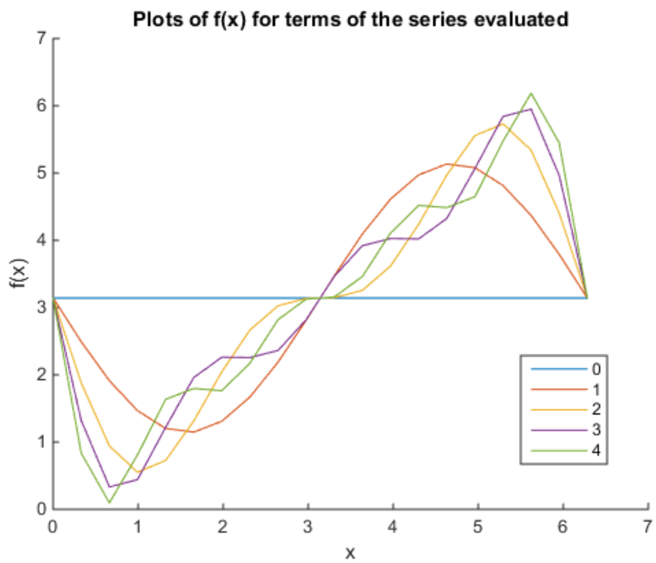
$$a_k = \frac{1}{\pi}\int_0^{2\pi} x\cos(kx)dx = \frac{1}{\pi}\left[\frac{\cos(kx) + kx\sin(kx)}{k^2}\right]_0^{2\pi} = \frac{1 + 0 - 1 - 0}{\pi k^2} = 0$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} x dx = \frac{1}{2\pi}\frac{1}{2}(2\pi)^2 - 0 = \pi$$

Thus, the Fourier series for $f(x) = x$ on $(0, 2\pi)$ is

$$x = \pi + \frac{-2}{1}\sin(x) + \frac{-2}{2}\sin(2x) + \frac{-2}{3}\sin(3x) + \frac{-2}{4}\sin(4x) + \frac{-2}{5}\sin(5x) + \cdots$$

(b) Plot the first few terms of the series in (1) using the coefficients you found. (Hint: It is easy to do this in Matlab using a loop over k to add all the terms together.)



Note that, away from the end points, they are converging to the function $f(x) = x$, as expected.

Problem 3. Show that the quadrature

$$\int_0^\infty e^{-x} f(x)dx \approx \frac{2+\sqrt{2}}{4} f(2-\sqrt{2}) + \frac{2-\sqrt{2}}{4} f(2+\sqrt{2})$$

has algebraic degree of accuracy 3.

Proof. Since the quadrature is linear in f (as most quadratures are), we do not need to take an arbitrary polynomial ax^3+bx^2+cx+d. It is easier to take $f(x) = 1$, $f(x) = x$, $f(x) = x^2$, and $f(x) = x^3$. Using calculus (namely, integration by parts), one can show:

$$\int_0^\infty e^{-x} 1 dx = 1 = \frac{2+\sqrt{2}}{4} 1 + \frac{2-\sqrt{2}}{4} 1$$

$$\int_0^\infty e^{-x} x dx = 1 = \frac{2+\sqrt{2}}{4}(2-\sqrt{2}) + \frac{2-\sqrt{2}}{4}(2+\sqrt{2})$$

$$\int_0^\infty e^{-x} x^2 dx = 2 = \frac{2+\sqrt{2}}{4}(2-\sqrt{2})^2 + \frac{2-\sqrt{2}}{4}(2+\sqrt{2})^2$$

$$\int_0^\infty e^{-x} x^3 dx = 6 = \frac{2+\sqrt{2}}{4}(2-\sqrt{2})^3 + \frac{2-\sqrt{2}}{4}(2+\sqrt{2})^3$$

$$\int_0^\infty e^{-x} x^4 dx = 24 \neq 20 = \frac{2+\sqrt{2}}{4}(2-\sqrt{2})^4 + \frac{2-\sqrt{2}}{4}(2+\sqrt{2})^4$$

So the quadrature is 3^{rd}-order, but not 4^{th}-order. This means it is exact for cubics, but not for 4^{th}-degree polynomials. □

Problem 4. Find the nodes and the coefficients of the Gauss quadrature with two nodes for evaluating the integral

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}dx.$$

Solution. We need to find w_1, w_2, x_1, and x_2 such that the expression

$$w_1 f(x_1) + w_2 f(x_2)$$

is exactly equal to the integral when $f(x) = 1$, $f(x) = x$, $f(x) = x^2$, and $f(x) = x^3$ (Gaussian quadrature on n points is always exact to degree $2n-1$ and no higher. Since we have two nodes, $n = 2$, and $2n-1 = 3$, so we will get a 3$^{\text{rd}}$-order method.)

A little calculus (trig-substitution) gives the following table.

$f(x)$	$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}dx$	Resulting Equation
1	π	$\pi = w_1 + w_2$
x	0	$0 = w_1x_1 + w_2x_2$
x^2	$\frac{\pi}{2}$	$\frac{\pi}{2} = w_1x_1^2 + w_2x_2^2$
x^3	0	$0 = w_1x_1^3 + w_2x_2^3$

In fact, for $f(x) = x$ and $f(x) = x^3$, the integrands are odd functions, so the integral is zero because we are integrating over a symmetric interval.

Thus, $\pi = w_1 + w_2$ implies $w_1 = \pi - w_2$. Further, $0 = w_1x_1 + w_2x_2$ implies $0 = (\pi - w_2)x_1 + w_2x$ and $w_2 = \frac{\pi x_1}{x_1 - x_2}$. Observe,

$$\begin{aligned}
\frac{\pi}{2} &= w_1x_1^2 + w_2x_2^2 \\
&= (\pi - \frac{\pi x_1}{x_1 - x_2})x_1^2 + \frac{\pi x_1}{x_1 - x_2}x_2^2 \\
\Rightarrow x_2 &= \frac{-1}{2x_1}
\end{aligned}$$

Observe,

$$\begin{aligned}
0 &= w_1x_1^3 + w_2x_2^3 \\
&= \left(\pi - \frac{\pi x_1}{x_1 + \frac{1}{2x_1}}\right)x_1^3 - \left(\frac{\pi x_1}{x_1 + \frac{1}{2x_1}}\right)\frac{1}{8x_1^3} \\
\Rightarrow x_1 &= \pm\frac{1}{\sqrt{2}}
\end{aligned}$$

Thus $x_1 = \frac{1}{\sqrt{2}}, x_2 = \frac{-\sqrt{2}}{2}, w_1 = \frac{\pi}{2}$ and $w_2 = \frac{\pi}{2}$ or $x_1 = \frac{-1}{\sqrt{2}} = \frac{-\sqrt{2}}{2}, x_2 = \frac{\sqrt{2}}{2}, w_1 = \frac{\pi}{2}$ and $w_2 = \frac{\pi}{2}$. The resulting Gaussian quadrature is:

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}dx \approx \frac{\pi}{2}f\left(\frac{-\sqrt{2}}{2}\right) + \frac{\pi}{2}f\left(\frac{\sqrt{2}}{2}\right)$$

It is **exact for polynomials of degree 3 or less**, but not for general polynomials of degree 4 or higher.