MATH 447/847 - Numerical Analysis
Homework #5
Iterative Methods and Gershgorin Disks

Note: All the problems here, except for maybe 2(a), are very short, so look for simple solutions, rather than complicated ones!

Problem 0. Read pages 186-188 in Trefethen and Bau (read all of Chapter 24 if you want a good review of eigenvalues, diagonalizability, and geometric/algebraic multiplicity).

Problem 1 Given a square matrix $A = (a_{ij})_{i,j=1}^n$, let us define its **Gerschgorin disks** for $i = 1, \ldots, n$ by:

$$D_i = \left\{ z \in \mathbb{C} : |z - a_{ii}| \leq \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \right\}$$

These are disks in the complex plane $\mathbb{C}$, which are centered at the diagonal entries a_{ii}, and whose radius is the sum of the absolute values of the off-diagonal entries in the i^{th} row. They are very important tools in numerical analysis.

Draw a picture (on the same plane) of the Gershgorin disks D_1, D_2, D_3 for the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}.$$

Problem 2(a) Do problem 24.2(a) and 24.2(c) in Trefethen and Bau.

Problem 2(b) Gershgorin's Theorem (sometimes called Gershgorin's Localization Theorem) says that all the eigenvalues of a matrix must live in the Gershgorin disks. Without computing the eigenvalues of the matrix (it's not worth your time), tell whether the matrix in Problem 1 could have an eigenvalue greater than 6, or less than zero, based on your picture.

Problem 3 Consider trying to solve the problem $A\mathbf{x} = \mathbf{b}$ by an iteration method $\mathbf{x}^{k+1} = G\mathbf{x}^k + b$ for some G. If we write $A = D - L - U$, where D is a diagonal matrix made from the diagonal entries of A, $-L$ are the lower entries of A, and $-U$ are the upper entries of A then the Jacobi method is to choose $G = D^{-1}(L+U)$. That is, given $A = (a_{ij})_{i,j=1}^n$, set G to be

$$G = \begin{bmatrix} 0 & -\frac{a_{12}}{a_{11}} & -\frac{a_{13}}{a_{11}} & \cdots & -\frac{a_{1,n}}{a_{11}} \\ -\frac{a_{21}}{a_{22}} & 0 & -\frac{a_{23}}{a_{22}} & \cdots & -\frac{a_{2,n}}{a_{22}} \\ -\frac{a_{31}}{a_{33}} & -\frac{a_{32}}{a_{33}} & 0 & \cdots & -\frac{a_{3,n}}{a_{33}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{a_{n1}}{a_{nn}} & -\frac{a_{n2}}{a_{nn}} & -\frac{a_{n3}}{a_{nn}} & \cdots & 0 \end{bmatrix}.$$

Recall that a strictly-diagonally-dominant (SDD) matrix $A = (a_{ij})_{i,j=1}^n$, is a matrix such that

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \text{ for all } j = 1, \dots, n.$$

Prove that if A is SDD, then the Jacobi method converges. [Hint: Use the Fundamental Theorem of Iterative Methods and Gershgorin's Theorem, which together make the proof just a couple of lines.]

Problem 4(a) Consider Richard's iteration method with the scaling parameter $\tau > 0$ ($A^{-1} \approx B = \tau I$, $Q = B^{-1} = \frac{1}{\tau} I$), so that the iteration $\mathbf{x}^{k+1} = (I - BA)\mathbf{x}^k + Bb$ is just

$$\mathbf{x}^{k+1} = (I - \tau A)\mathbf{x}^k + \tau b,$$

so that $G = (I - \tau A)$. Recall the error equation $\mathbf{e}^{k+1} = G\mathbf{e}^k$. Show that

$$\|\mathbf{e}^{k+1}\|_2 \leq \left(\max_i |1 - \tau\lambda_i| \right) \|\mathbf{e}^k\|_2$$

where λ_i are the eigenvalues of A.

Problem 4(b) Let us order the eigenvalues so that $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Show that

$$\max_i |1 - \tau\lambda_i| = \max\{|1 - \tau\lambda_1|, |1 - \tau\lambda_n|\}$$

and that this quantity is **smallest** when $1 - \tau\lambda_1 = -1 + \tau\lambda_n$. (Hint: Don't use calculus to find the min of the max, it is ugly. Instead, just draw pictures of $|1 - \tau\lambda_i|$.) Also, solve for τ in this case.