MATH 447/847 - Numerical Analysis
Homework #6
FFT and Numerical Quadrature

Problem 1 Let $i = \sqrt{-1}$. Show that, for any integers m and n,

$$\int_0^{2\pi} e^{inx}e^{-imx}\,dx = \begin{cases} 2\pi & \text{if } m = n, \\ 0 & \text{if } m \neq n, \end{cases}$$

$$\int_0^{2\pi} \cos(nx)\cos(mx)\,dx = \begin{cases} \pi & \text{if } m = n \neq 0, \\ 0 & \text{if } m \neq n, \end{cases}$$

$$\int_0^{2\pi} \sin(nx)\cos(mx)\,dx = 0,$$

$$\int_0^{2\pi} \sin(nx)\sin(mx)\,dx = \begin{cases} \pi & \text{if } m = n \neq 0, \\ 0 & \text{if } m \neq n. \end{cases}$$

Time saver: Do the first two (use a trig identity on the second one), then expand the exponents in the first one with Euler's formula, ($e^{i\theta} = \cos(\theta) + i\sin(\theta)$), to get the rest.

Problem 2 Consider the "Continuous to Discrete" Fourier transform, given by the relations

$$f(x) = a_0 + \sum_{k=1}^{\infty}(a_k\cos(kx) + b_k\sin(kx)) \tag{1}$$

(a) Find the Fourier coefficients a_0, a_k, and b_k in the case where $f(x) = x$. (Hint: Use the relationships in Problem 1 on (1) to isolate the coefficents. Then integrate by parts.) The coefficients will be numbers that depend only on k.

(b) Plot the first few terms of the series in (1) using the coefficients you found. (Hint: It is easy to do this in Matlab using a loop over k to add all the terms together.)

Problem 3 Show that the quadrature

$$\int_0^{\infty} e^{-x}f(x)dx \approx \frac{2+\sqrt{2}}{4}f(2-\sqrt{2}) + \frac{2-\sqrt{2}}{4}f(2+\sqrt{2})$$

has algebraic degree of accuracy 3.

Problem 4 Find the nodes and the coefficients of the Gauss quadrature with two nodes for evaluating the integral

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}dx.$$