MATH 447/847 - Numerical Analysis
Homework #7
FOR PRACTICE ONLY—NOT TO BE TURNED IN
Numerical Methods for ODEs

Prob. 1 For the above initial value problem consider the following (in general, implicit) Runge-Kutta method, where:

$$\begin{aligned}\Phi(t,x;h) &= c_1k_1 + c_2k_2\\ k_1 &= f(t+\alpha_1 h, x+\beta_{11}hk_1)\\ k_2 &= f(t+\alpha_2 h, x+\beta_{21}hk_1+\beta_{22}hk_2).\end{aligned}$$

(a) Find the conditions that the coefficients $c_i, \alpha_i, \beta_{ij}$ need to satisfy so that the **explicit** method (i.e. $\beta_{11} = \beta_{22} = 0$ in the Butcher Tableau) is of second order. (Hint: use Taylor expansion, as demonstrated in class). Give at least one set of coefficients that satisfy these conditions.

(b) Find the conditions that the coefficients a, α, β need to satisfy so that the **implicit** method (i.e. $\beta_{11} \neq 0$ and $\beta_{22} \neq 0$) is of second order.

Prob. 2 Derive an explicit multi-step method of order four (Adams-Bashforth four step method) that uses integration in the interval (t_n, t_{n+1}). (Hint: For $y' = f(t,y)$, consider the integration

$$y(t_{n+1}) - y(t_n) = \int_{t_n}^{t_{n+1}} f(t, y(t))\, dt$$

and approximate $g(t) := f(t, y(t))$ by a sufficiently high-order polynomial based on previous time steps. This was done in class to derive Adams-Bashforth-2 by using a first-order polynomial based at t_n and t_{n-1}.)

Prob. 3 Solve the following linear difference equation problem:

$$\eta_n - 5\eta_{n-1} + 4\eta_{n-2} = 0, \qquad \eta_0 = 1, \quad \eta_1 = 0.$$

Prob. 4 Consider the Cauchy problem $x' = f(t,x)$, $x(0) = x_0$ and its approximation by the following two-step method:

$$\eta_n - \eta_{n-2} = 3hf_n - hf_{n-2},\ n = 2, 3, ...,$$

where $\eta_0 = x_0$, η_1 is given (or computed somehow), and $f_n = f(t_n, \eta_n)$.
(a) Check whether the above method is **consistent** (as $h \to 0$) and **stable** (for sufficiently small h) (Hint: Use the theorems involving the polynomials p and q, discussed in class on Wednesday, April 29.)
(b) Estimate the **local truncation error** and find the order of the method;
(c) Check whether this method is **A-stable**.

Prob. 5 Study the Dahlquist barriers and the Lax Equivalence Theorem.