FULL NAME: Key Exam 2

MATH 447, Numerical Analysis Dr. Adam Larios No calculators or notes

Answers without full, proper justification will not receive full credit.

For a square matrix $A = (a_{ij})_{i,j=1}^n$, its **Gerschgorin disks** for $i = 1, \ldots, n$ are:

$$D_i = \left\{ z \in \mathbb{C} : |z - a_{ii}| \le \sum_{\substack{j=1 \\ j \ne i}}^{n} |a_{ij}| \right\}$$

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1. **(6 points)** To solve $A\mathbf{x} = \mathbf{b}$ by an iterative method using splitting, one often finds a non-singular matrix Q related to A somehow, and then uses the iterative formula

$$Q\mathbf{x}^{k+1} = (Q - A)\mathbf{x}^k + \mathbf{b}$$

Give a condition which guarantees that the iteration converges for any initial guess.

$\rho(Q^{-1}(Q-A)) < 1$ (by Fundamental Theorem of Iterative Methods, F.T.I.M.)

2. **(10 points)** Let $A = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$. Write A^{-1} as a linear combination of I, A, and A^2. (**Hint:** use a theorem we discussed in class.)

By the Cayley Hamilton theorem, every matrix satisfies its own characteristic polynomial.

Note that $\det(A - \lambda I) = (1-\lambda)^3 = -\lambda^3 + 3\lambda^2 - 3\lambda + 1$.

Thus, $-A^3 + 3A^2 - 3A + I = 0 \Rightarrow -A^2 + 3A - 3I + A^{-1} = 0$

$\Rightarrow \boxed{A^{-1} = A^2 - 3A + 3I}$

3. **(15 points)** The SOR method of iteration has an iteration matrix G given by

$$G = (I - \omega D^{-1}L)^{-1}[(1-\omega)I - \omega D^{-1}U]$$

where ω is a real number, L is *strictly* lower-triangular, and U is *strictly* upper-triangular, and D is a diagonal matrix. Show that if $0 < \omega < 2$, then SOR converges, and it diverges otherwise. **Hint:** Use the fact that the determinant of a matrix is the product of its eigenvalues, and $\det(AB) = \det(A)\det(B)$. **Note:** This problem does not require anything complicated.

By F.T.I.M., only need to show $\rho(G) < 1$, i.e. all eigenvalues λ satisfies $|\lambda| < 1$. By hint, $\det(G) = \lambda_1\lambda_2 \cdots \lambda_n$.

Since $\det(AB) = \det(A)\det(B)$, we have $\det(A^{-1}) = \det(A)^{-1}$ and

$|\lambda_1 \cdots \lambda_n| = |\det(G)| = \left|\det(I - \omega \underbrace{D^{-1}L}_{\text{strictly lower}})\right|^{-1} \left|\det((1-\omega)I - \omega \underbrace{D^{-1}U}_{\text{strictly upper}})\right|$

$= |\overbrace{1 \cdot 1 \cdot 1 \cdots 1}^{n \text{ times}}|^{-1} |(1-\omega)^n| = |1-\omega|^n$

So to have all eigenvalues < 1, need $|1-\omega|^n < 1$, or $|1-\omega| < 1$, or $0 < \omega < 2$.

4. **(4 points)** Professor Axby is trying to solve the system $A\mathbf{x} = \mathbf{b}$ by the conjugate gradient method where A is SPD. She finds the condition number to be $\kappa(A) \approx 10^6$. Being a skilled numerical analyst, what does she do first?

   (a) Find a preconditioner to decrease the condition number.

   (b) Find a preconditioner to increase the condition number. ← makes things worse

   (c) Nothing, since A is already SPD, and a preconditioner could ruin that. ← Just use SPD with a different inner-product based on the preconditioner.

   (d) Use GMRES instead. ← slower, so use CG instead

5. Consider the system $A\mathbf{x} = \mathbf{b}$ given by

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

   (a) **(8 points)** Find all the Krylov subspaces K_i associated with this system.

$b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\quad Ab = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\quad A^2 b = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, so only two.

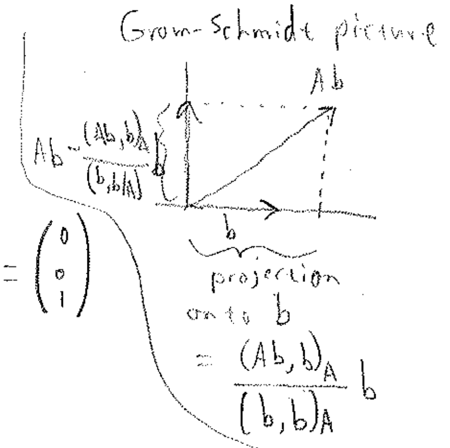
$K_1 = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}$, $\quad K_2 = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right\}$

   (b) **(12 points)** Find A-orthonormal vectors $\mathbf{q}_1$ and $\mathbf{q}_2$ such that $\text{span}\{\mathbf{q}_1\} = K_1$ and $\text{span}\{\mathbf{q}_1, \mathbf{q}_2\} = K_2$. Hint: Use (classical) Gram-Schmidt with a different inner-product.

- $q_1 = \dfrac{b}{\|b\|_A} = \dfrac{b}{\sqrt{(Ab,b)}} = \dfrac{b}{\sqrt{1^2+1^2+0^2}} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$

- $Ab - \dfrac{(Ab,b)_A}{(b,b)_A} b = Ab - \dfrac{(A^2b,b)}{(Ab,b)} b = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \dfrac{\sqrt{2}}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

- $q_2 = \dfrac{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}{\left\|\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\|_A}$ ← divide by zero, cannot normalize



6. **(10 points)** Let A be a symmetric $m \times m$ SDD (strictly diagonally dominant) matrix such that every diagonal element is positive. Show that all of its eigenvalues are positive.

SDD means $|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{m} |a_{ij}|$. By Gershgorin's Theorem, all eigenvalues live in Gershgorin disks, so $|\lambda - a_{ii}| \le \sum_{\substack{j=1 \\ j\neq i}}^{m} |a_{ij}| \overset{\text{SDD}}{<} |a_{ii}| = a_{ii}$ ← diagonals are assumed to be positive.

Thus $|\lambda - a_{ii}| < a_{ii}$, or $\underbrace{-a_{ii} < \lambda - a_{ii}}_{0 = a_{ii} - a_{ii} < \lambda} < a_{ii}$

7. **(5 points)** Professor Axby is trying to solve the system $A\mathbf{x} = \mathbf{b}$ by the conjugate gradient method where A is SPD. She finds the condition number to be $\kappa(A) \approx 10^6$. Being a skilled numerical analyst, what does she do first?

   (a) Find a preconditioner to decrease the condition number.

   (b) Find a preconditioner to increase the condition number.

   (c) Nothing, since A is already SPD, and a preconditioner could ruin that.

   (d) Use GMRES instead.

Oops! Duplicate question.

8. **(15 points)** The Jacobi iteration for solving $A\mathbf{x} = \mathbf{B}$ is given by $\mathbf{x}^{k+1} = G\mathbf{x}^k + D^{-1}b$, where G is given in terms of the entries of A by:

$$G = \begin{bmatrix} 0 & -\frac{a_{12}}{a_{11}} & -\frac{a_{13}}{a_{11}} & \cdots & -\frac{a_{1,n}}{a_{11}} \\ -\frac{a_{21}}{a_{22}} & 0 & -\frac{a_{23}}{a_{22}} & \cdots & -\frac{a_{2,n}}{a_{22}} \\ -\frac{a_{31}}{a_{33}} & -\frac{a_{32}}{a_{33}} & 0 & \cdots & -\frac{a_{3,n}}{a_{33}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{a_{n1}}{a_{nn}} & -\frac{a_{n2}}{a_{nn}} & -\frac{a_{n3}}{a_{nn}} & \cdots & 0 \end{bmatrix}.$$

Prove that if A is SDD (strictly diagonally dominant), then the Jacobi method converges.

This was on HW#5.

F.T.I.M. $\Rightarrow$ only need to show $\rho(G) < 1$.

Gershgorin $\Rightarrow$ All eigenvalues λ are in Gershgorin disks, that is,

$$|\lambda - 0| \leq \sum_{\substack{j=1 \\ j\neq i}}^{n} \left| -\frac{a_{ij}}{a_{ii}} \right| = \frac{1}{|a_{ii}|} \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| \overset{SDD}{<} \frac{1}{|a_{ii}|} \cdot |a_{ii}| = 1$$

So $|\lambda| < 1$ for all eigenvalues, so iteration converges.

9. **(15 points)** Let $A \in \mathbb{R}^{m\times m}$ be an SPD (symmetric positive-definite) matrix and let $\mathbf{b} \in \mathbb{R}^m$ be given. Consider the real-valued function

$$J(\mathbf{y}) = \tfrac{1}{2}\mathbf{y}^T A\mathbf{y} - \mathbf{y}^T\mathbf{b} = \tfrac{1}{2}(A\mathbf{y}, \mathbf{y}) - (\mathbf{b}, \mathbf{y})$$

Let $\mathbf{y}$ and $\mathbf{u}$ be a non-zero vectors, and define $\varphi(\alpha) = J(\mathbf{y} + \alpha\mathbf{u})$. Find the value of α that minimizes φ.

$$\boxed{\begin{aligned} &\text{SPD} \Rightarrow \\ &(Au, y) = (u, Ay) \\ &= (Ay, u) \end{aligned}}$$

$$\varphi(\alpha) = J(y + \alpha u) = \frac{1}{2}\left(A(y+\alpha u), y + \alpha u\right) - (b, y + \alpha u)$$

$$= \frac{1}{2}\left[A(y,y) + \alpha(Au, y) + \alpha(Ay, u) + \alpha^2(u,u)\right] - (b,y) - \alpha(b,u)$$

$$= \frac{1}{2}(Ay, y) + \cancel{\tfrac{2}{2}}\alpha(Ay, u) + \frac{\alpha^2}{2}\|u\|^2 - (b,y) - \alpha(b,u)$$

Minimum when $\varphi'(\alpha) = 0$:

$$0 = \varphi'(\alpha) = (Ay, u) + \alpha\|u\|^2 - (b, u)$$

$$\Rightarrow \alpha = \frac{(b,u) - (Ay,u)}{\|u\|^2} = \frac{(b - Ay, u)}{\|u\|^2} = \frac{(\overset{\text{residual}}{r}, u)}{(u,u)}$$