1. Find the maximum and minimum of $f(x) = \dfrac{x(x-1)}{x^2+3x+3}$ on the interval $[0, 6]$ and then on the real line.

   Solution
   First, we compute the derivative:

   $$f'(x) = \frac{4x^2+6x-3}{(x^2+3x+3)^2}$$

   Solving $f'(x) = 0$ means solving $4x + 6x - 3$, which, by means of the quadratic formula, gives $-3/4 \pm \sqrt{21}/4$. As $-3/4 - \sqrt{21}/4 < 0$, for the interval $[0, 6]$, we can ignore this critical point. We evaluate f at 0, $-3/4 + \sqrt{21}/4$, and 6, giving 0, $-.0551$, and $10/19$. So the maximum of f on $[0, 6]$ is $10/19$ at 6 and the minimum is about -0.0551 at $-3/4 + \sqrt{21}/4$.

   On the whole real line, we compute $f(-3/4 - \sqrt{21}/4) = 6.0551$. Notice that

   $$\lim_{x\to\pm\infty} f(x) = 1$$

   and since f attains values smaller and larger than 1 at its critical points, the values at these critical points are the extrema of the function. Thus, the maximum of f on $\mathbf{R}$ is 6.0551 at $-3/4 - \sqrt{21}/4$ and the minimum of f on $\mathbf{R}$ is $-.0551$ at $-3/4 + \sqrt{21}/4$.

2. Compute the gradient $\nabla f(\mathbf{x})$ of the function $f(x_1, x_2, x_3) = 3x_1^2 + 4x_1x_2 + x_2\sin(x_3)$.

   Solution
   We compute

   $$\begin{aligned}\nabla f(\mathbf{x}) &= \left(\frac{\partial f}{\partial x_1}(\mathbf{x}), \frac{\partial f}{\partial x_2}(\mathbf{x}), \frac{\partial f}{\partial x_3}(\mathbf{x})\right) \\ &= \left(6x_1 + 4x_2, 4x_1 + \sin(x_3), x_2\cos(x_3)\right).\end{aligned}$$

3. Solve the following linear system of equations. If the system is consistent, write the solution in vector form.

   $$\left\{\begin{array}{rrrcr} x & -y & +z & = & 0 \\ -x & +3y & +z & = & 5 \\ 3x & +y & +7z & = & 10 \end{array}\right.$$

Solution

We first compute the reduced row echelon form of the augmented matrix A :

$$A = \begin{bmatrix} 1 & -1 & 1 & 0 \\ -1 & 3 & 1 & 5 \\ 3 & 1 & 7 & 10 \end{bmatrix}$$

Since rank(A)=2¡# of variables , the system is consistent, and the solution has 1 free variable and hence an infinite number of solutions. From the RREF of A , we have that

$$\begin{cases} x \quad +2z = \frac{5}{2} \\ \quad y \quad +z = \frac{5}{2} \\ \qquad 0 = 0 \end{cases}$$

Hence,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2z + \frac{5}{2} \\ -z + \frac{5}{2} \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} z + \begin{bmatrix} \frac{5}{2} \\ \frac{5}{2} \\ 0 \end{bmatrix}.$$

4. For each matrix below, determine if it is invertible. If so, compute its inverse.

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 3 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 3 \\ -1 & -1 \end{bmatrix}, \qquad C = \begin{bmatrix} 2 & 3 & 1 \\ 2 & 1 & 3 \\ 1 & 2 & 0 \end{bmatrix}.$$

Solution

A is not invertible since it is not square.

We find the reduced row echelon form of

$$[B \mid I_{2\times 2}] : \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ -1 & -1 & 0 & 1 \end{array}\right]$$

Since the reduced row echelon form of B is I , B is invertible and

$$B^{-1} = \begin{bmatrix} -\frac{1}{2} & -\frac{3}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

We find the reduced row echelon form of C :

$$\begin{bmatrix} 2 & 3 & 1 \\ 2 & 1 & 3 \\ 1 & 2 & 0 \end{bmatrix}.$$

Since the reduced row echelon form of C is not the identity $I_{3\times 3}$, C is not invertible.

5. Let
$$A = \begin{bmatrix} 1 & 3 \\ -2 & 6 \end{bmatrix}.$$
Find all of the eigenvalues of A and bases for the corresponding eigenspaces.

<u>**Solution**</u>
The characteristic polynomial of A is
$$\det \begin{bmatrix} 1-\lambda & 3 \\ -2 & 6-\lambda \end{bmatrix} = (1-\lambda)(6-\lambda)+6 = \lambda^2 - 7\lambda + 12 = (\lambda-3)(\lambda-4).$$
The solutions of $\det(A-\lambda I) = 0$ are the eigenvalues of A. Hence, the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = 4$.

To find bases for the corresponding eigenspaces, we find a basis for the nullspace of $\det(A-\lambda I)$ for each eigenvalue λ. For $\lambda_1 = 3$,
$$A - 3I = \begin{bmatrix} -2 & 3 \\ -2 & 3 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & -3/2 \\ 0 & 0 \end{bmatrix}, \quad \text{so} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 1 \end{bmatrix} x_2$$
and so
$$E_3 = \text{span}\left\{ \begin{bmatrix} 3/2 \\ 1 \end{bmatrix} \right\}.$$
For $\lambda_2 = 4$,
$$A - 4I = \begin{bmatrix} -3 & 3 \\ -2 & 2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \quad \text{so} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} x_2,$$
and so
$$E_4 = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$

6. Are the following vectors linearly independent? Prove your answer.
$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 7 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

<u>Solution</u>

Calling the three vectors $\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \mathbf{v}^{(3)}$, we wish to know whether scalars a_1, a_2, a_3 exist, not all zero, such that $a_1\mathbf{v}^{(1)} + a_2\mathbf{v}^{(2)} + a_3\mathbf{v}^{(3)} = \mathbf{0}$. If such scalars exist, there exists a linear dependence among the vectors and they are linearly dependent. If no such scalars exist, then the vectors are linearly independent.

We thus wish to determine the number of solutions to the linear system $a_1\mathbf{v}^{(1)} + a_2\mathbf{v}^{(2)} + a_3\mathbf{v}^{(3)} = \mathbf{0}$. Putting the three vectors as columns into a matrix A and computing the RREF, we see that

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 2 & 3 & 1 & 0 \\ 3 & 2 & 1 & 0 \\ 7 & 2 & 1 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and hence the only solution is $a_1 = a_2 = a_3 = 0$. Thus, the vectors are linearly independent.