1. Finish typing the following Matlab programs from the Matlab Intro on Blackboard: `factorial.m`, `fibonacci.m`, `choose.m`. Print out your three `*.m` files and attach them to this homework. (Don't copy/paste into Word or anything, just print from Matlab.)

   Solution.
   See the Matlab Intro file on Blackboard.

2. Read sections 1,1, 1.5, 1.6, and 1.7 in the book.

3. Do problem 7.1 (page 22) and problem 7.1 (page 27). (Short answers please.)
   Solution to problem 7.1 (page 22)

   Answers may vary slightly in formulation. Here is one possible formulation. Let c_{ij} be the cost of an aircraft of type i on arc j. Let x_{ij} be the number of aircraft of type i on arc j, and let X be the matrix with entries x_{ij}. Let Q be the arc-node index matrix. Let M_i be the total number of available aircraft of type i.

$$\begin{aligned} \min \quad & \sum_j \sum_i c_{ij} x_{ij} \\ \text{subject to:} \quad & \sum_i x_{ij} = 1 \\ & XQ = 0 \\ & \sum_j x_{ij} \leq M_i \\ & x_{ij} = 0 \text{ or } 1 \end{aligned}$$

   Solution to problem 7.1 (page 27)
   Answers may vary slightly in formulation. Here is one possible formulation. (See the book for the notation.)

$$\begin{aligned} \max \quad & \mathbf{r}^T \mathbf{x} - \alpha \mathbf{x}^T V \mathbf{x} + r_0 x_0 \\ \text{subject to:} \quad & x_0 + \sum_{i \neq 0} \mathbf{x}_i = 1 \\ & x_i \geq 0 \end{aligned}$$

   Where r_0 is the rate associated with asset 0, and x_0 is the proportion of the investment to be in vested in assent 0.

4. A factory has 9 lathes and 4 grinders. Each machine runs for 40 hours per week. The machines are used to make three different products. Each unit of product 1 requires 2 hours of time on a grinder machine, each unit of product 2 requires 4 hours on a lathe, and each unit of product 3 requires 5 units on a lathe and 3 units on a grinder. The products also have monetary costs of 25, 10, and 15, respectively. The sale price per unit depends on the supply, and is given by $p_1(x_1) = 20 + 50x_1^{-1/2}$, $p_2(x_2) = 15 + 40x_2^{-1/4}$, and $p_3(x_3) = 35 + 100x_3^{-1/3}$, where x_j is the number of units of product j per week.

   (a) What key properties do the functions p_j have? How realistic are these?
   Solution.
   The easiest way to see this is to plot the functions. First, notice that $\lim_{x\to 0^+} p_j(x) = \infty$, and selling no items shouldn't lead to infinite profit, but then, we can just modify the functions to set $p_j(0) = 0$, and note that they should only be defined on the non-negative integers. Next, note that as you make more products, the cost decreases. On the other hand, $\lim_{x\to 0^+} p_1(x) = 20$, so making more products eventually does not affect the individual price much. Similar comments hold for p_2 and p_3.

   (b) The company wants to maximize its weekly profit. Construct the appropriate objective function.
   Solution.
   If x_i items are sold at a price $p_i(x_i)$, then the revenue is $x_i p_i(x_i)$. Denote the costs of each item by c_i. Since Profit = Revenue - Cost, the profit from each type of product is $x_i p_i(x_i) - c_i x_i$. The total profit is

$$\begin{aligned}
&\sum_{i=1}^{3}(x_i p_i(x_i) - c_i x_i)\\
=&(x_1(20 + 50x_1^{-1/2}) - 25x_1) + (x_2(15 + 40x_2^{-1/4}) - 10x_2)\\
&+ (x_3(35 + 100x_3^{-1/3}) - 15x_3)
\end{aligned}$$

   This is the objective function; that is, the function to be maximized.

   (c) Construct appropriate constraints for the amounts of products made. There are a total of four constraints.
   Solution.
   For the lathe, we must have

$$4x_2 + 5x_3 \leq 9 \cdot 40$$

For the grinder, we must have

$$2x_1 + 3x_3 \leq 4 \cdot 40$$

We also have the following two physical constraints:

$$x_1, x_2, x_3 \geq 0,$$
$$x_1, x_2, x_3 \text{ must be integers.}$$

5. Suppose we have a collection of $n > 3$ data points (x_j, y_j), and we wish to find the quadratic polynomial function $f(x) = a + bx + cx^2$ that minimizes the residual sum of squares $R(a, b, c) = \sum_{j=1}^{n}(y_j - f(x_j))^2$. Let $\mathbf{u}$ be the column vector with components a, b, and c. Use the fact that the minimizer must occur at a point where the first partial derivatives of R are all 0 to construct a matrix $\mathbf{M}$ and a vector $\mathbf{v}$ such that $\mathbf{v}$ is determined by the equation $\mathbf{Mu} = \mathbf{v}$.[1]

Solution. Write

$$R = R(a, b, c) = \sum_{j=1}^{n}(y_j - a - bx_j - cx_j^2)^2$$

Note that R is quadratic in a, b, and c, and its graph will be a convex paraboloid, so in particular, it has a unique minimum. At the minimum, we must have $\nabla R = \mathbf{0}$. Using the chain-rule, we find

$$0 = \frac{\partial R}{\partial a} = \sum_{j=1}^{n} 2(y_j - a - bx_j - cx_j^2)(-1)$$
$$0 = \frac{\partial R}{\partial b} = \sum_{j=1}^{n} 2(y_j - a - bx_j - cx_j^2)(-x_j)$$
$$0 = \frac{\partial R}{\partial c} = \sum_{j=1}^{n} 2(y_j - a - bx_j - cx_j^2)(-x_j^2)$$

Rearranging these equations to put the unknown variables a, b, and c on one

[1] In general, the first partial derivatives all 0 is a necessary condition for a minimizer but not a sufficient condition. We will eventually learn a theorem that gives additional restrictions needed for the condition to be sufficient as well as necessary.

side, we find (after dividing by 2),

$$a\sum_{j=1}^{n} 1 + b\sum_{j=1}^{n} x_j + c\sum_{j=1}^{n} x_j^2 = \sum_{j=1}^{n} y_j$$
$$a\sum_{j=1}^{n} x_j + b\sum_{j=1}^{n} x_j^2 + c\sum_{j=1}^{n} x_j^3 = \sum_{j=1}^{n} y_j x_j$$
$$a\sum_{j=1}^{n} x_j^2 + b\sum_{j=1}^{n} x_j^3 + c\sum_{j=1}^{n} x_j^4 = \sum_{j=1}^{n} y_j x_j^2$$

Remember that the x_j and y_j are just given numbers. We can rewrite the above system of equations in matrix form as

$$\begin{bmatrix} \sum_{j=1}^{n} 1 & \sum_{j=1}^{n} x_j & \sum_{j=1}^{n} x_j^2 \\ \sum_{j=1}^{n} x_j & \sum_{j=1}^{n} x_j^2 & \sum_{j=1}^{n} x_j^3 \\ \sum_{j=1}^{n} x_j^2 & \sum_{j=1}^{n} x_j^3 & \sum_{j=1}^{n} x_j^4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{n} y_j \\ \sum_{j=1}^{n} y_j x_j \\ \sum_{j=1}^{n} y_j x_j^2 \end{bmatrix}$$

which is the form $\mathbf{Mu} = \mathbf{v}$ that we wanted. Note also that $\sum_{j=1}^{n} 1 = n$.