Math 433/833 **Assignment # 2** **Assigned: 2016.01.28**
Due: 2016.02.09

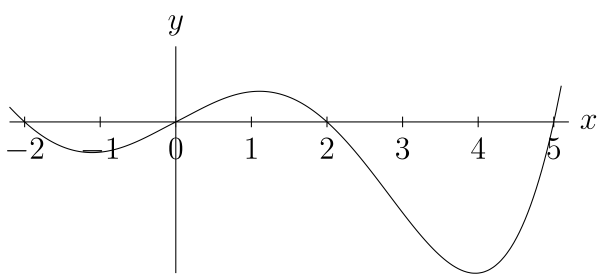
0. Read sections 2.1, 2.2, 2.3, 2.4, 2.5 in the book.

1. Problems:

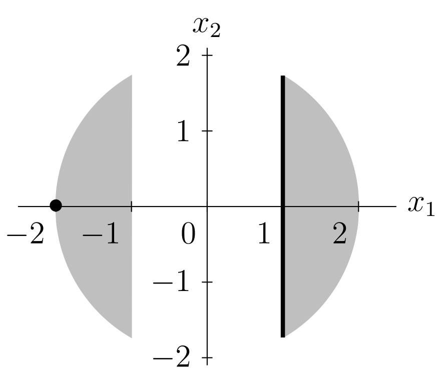
Page 47, #2.1 **Solution.** All the given points are feasible (plug them in, and see that they satisfy the inequalities). x_b, x_c, and x_e are on the boundary of the first constraint (the satisfy not just the inequality, but are actually equal). x_e is on the boundary of the second constraint. x_b, x_c, and x_d are on the boundary of the third constraint.

Page 47, #2.2 $f(x) = (x+1)x(x-2)(x-5) = x^4 - 6x^3 + 3x^2 + 10x$.

Solution. Stationary points at (roughly) $x = -1, 1, 4$. There are local minima at about $x = -1, 4$. There is a local maximum at about $x = 1$. There is no global maximum.



Page 47, #2.3 **Solution.** The feasible set is the shaded region below. The set of local minimizers is draw in black (the vertical line segment, and the points at $(-2, 0)$). The global minmimizer is the point $(-2, 0)$.



Page 52, #3.3 Prove that the set $S = \{\mathbf{x} : A\mathbf{x} \le \mathbf{b}\}$ is convex.

Solution. To show convexity, we need to take two arbitrary points in S, and show that any point on the line joining them is also in S.

Let $\mathbf{x}, \mathbf{y} \in S$ be arbitrary, and let $0 \leq \alpha \leq 1$ be arbitrary. We will show that $\mathbf{z}$, defined by $\mathbf{z} = \alpha\mathbf{x} + (1-\alpha)\mathbf{y} \in S$. We need to show that $A\mathbf{z} \leq \mathbf{b}$.

$$\begin{aligned} A\mathbf{z} &= A(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) \\ &= \alpha A\mathbf{x} + (1-\alpha)A\mathbf{y} && \text{(by properties of matrices)} \\ &\leq \alpha\mathbf{b} + (1-\alpha)\mathbf{b} && (\text{since } \mathbf{x}, \mathbf{y} \in S) \\ &= \mathbf{b} \end{aligned}$$

Thus $A\mathbf{z} \leq \mathbf{b}$, so $\mathbf{z} \in S$. Since $\mathbf{x}, \mathbf{y}, \alpha$ were arbitrary, S is convex.

Page 52, #3.12 Let $g_1, \ldots, g_m$ be concave functions on $\mathbb{R}^n$. Prove that the set $S = \{\mathbf{x} : g_i(\mathbf{x}) \geq 0, i = 1, \ldots, m\}$ is convex.

Solution. Let $\mathbf{x}, \mathbf{y} \in S$ be arbitrary, and let $0 \leq \alpha \leq 1$ be arbitrary. Since $\mathbf{x}, \mathbf{y} \in S$, we know that

$$\begin{aligned} g_i(\mathbf{x}) \geq 0 \text{ for each } i = 1, \ldots, m, \text{ and} \\ g_i(\mathbf{y}) \geq 0 \text{ for each } i = 1, \ldots, m. \end{aligned}$$

Since each g_i is concave, we also have, for each $i = 1, \ldots, m$,

$$\begin{aligned} g_i(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) &\geq \alpha g_i(\mathbf{x}) + (1-\alpha)g_i(\mathbf{y}) \\ &\geq \alpha 0 + (1-\alpha)0 && (\text{since } \mathbf{x}, \mathbf{y} \in S) \\ &= 0. \end{aligned}$$

Thus, $g_i(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) \geq 0$, so $\alpha\mathbf{x} + (1-\alpha)\mathbf{y} \in S$. Thus, S is convex.

Page 52, #3.13 Let f be a convex function on the convex set S. Prove that the level set $T = \{\mathbf{x} : f(\mathbf{x}) \leq k\}$ is convex for all real numbers k.

Solution. Let $\mathbf{x}, \mathbf{y} \in T$ be arbitrary, and let $0 \leq \alpha \leq 1$ be arbitrary. Since $\mathbf{x}, \mathbf{y} \in S$, we know that

$$\begin{aligned} f(\mathbf{x}) &\leq k, \text{ and} \\ f(\mathbf{y}) &\leq k. \end{aligned}$$

Since f is convex, we also have

$$\begin{aligned} f(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) &\leq \alpha f(\mathbf{x}) + (1-\alpha)f(\mathbf{y}) \\ &\leq \alpha k + (1-\alpha)k && (\text{since } \mathbf{x}, \mathbf{y} \in T) \\ &= k. \end{aligned}$$

Thus, $f(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) \leq k$, so $\alpha\mathbf{x} + (1-\alpha)\mathbf{y} \in T$. Thus, T is convex.

Page 62, #5.1(v) [This problem was retracted, and will not be graded.]

Page 62, #5.2 Consider the sequence defined by $x_0 = a > 0$, and

$$x_{k+1} = \tfrac{1}{2}\left(x_k + \frac{a}{x_k}\right). \tag{1}$$

Prove that this sequence converges to $x_* = \sqrt{a}$.

Fun fact: This is exactly Newton's method used for computing $\sqrt{a}$, that is, finding a positive zero of $f(x) = x^2 - a$, since for Newton's method,

$$\begin{aligned} x_{k+1} &= x_k - \frac{f(x_k)}{f'(x_k)} \\ &= x_k - \frac{x_k^2 - a}{2x_k} \\ &= x_k - \frac{x_k^2}{2x_k} + \frac{a}{2x_k} \\ &= \tfrac{1}{2}\left(x_k + \frac{a}{x_k}\right). \end{aligned}$$

This method of finding square roots is also called the Babylonian method, as well as "Hero's method," after Hero of Alexandria.

Solution. (Note: the case $a = 2$ was already shown in class.) First, note that since $x_0 = a > 0$, we must always have $x_{k+1} > 0$, since $x_{k+1} = \frac{1}{2}\left(x_k + \frac{a}{x_k}\right) > 0$. (Note: This does not imply that the limit is positive.)

Next, *assume* that the sequence converges (later, we will show convergence). Then $x_k \to x_*$. We will show that $x_* = \sqrt{a}$. ***If*** it converges (to something other than zero), then we also have $x_{k+1} \to x_*$ and $a/x_{k+1} \to a/x_*$. Thus, taking the limit of the defining equation,

$$x_* = \tfrac{1}{2}\left(x_* + \frac{a}{x_*}\right)$$

After a little algebra, we find $x_*^2 = a$, so $x_* = \sqrt{a}$.

Next, denote $e_k = x_k - x_* = x_k - a$. Then, using equation (1), we find

$$\begin{aligned} e_{k+1} &= x_{k+1} - x_* \\ &= \tfrac{1}{2}\left(x_k + \frac{a}{x_k}\right) - \sqrt{a} \\ &= \tfrac{1}{2x_k}\left(x_k^2 + a - 2\sqrt{a}x_k\right) \\ &= \tfrac{1}{2x_k}\left(x_k - \sqrt{a}\right)^2 \\ &= \tfrac{1}{2x_k}e_k^2 \end{aligned}$$

Thus, $e_{k+1}/e_k^2 = \frac{1}{2x_k}$, so

$$\lim_{k\to\infty} \frac{e_{k+1}}{e_k^2} = \lim_{k\to\infty} \tfrac{1}{2x_k} = \tfrac{1}{2\sqrt{a}}.$$

This means that the convergence is quadratic with constant $C = \frac{1}{2\sqrt{a}}$. (Note: This means that finding square roots of larger numbers, the method will converge faster!)

Proof of convergence. (Not required for full credit.)

As already noted above, $x_k > 0$ for all k. Next, notice that, from the above calculation for the error, that

$$x_{k+1} - \sqrt{a} = e_{k+1} = \tfrac{1}{2x_k}e_k^2 \geq 0.$$

Thus, $x_{k+1} \geq \sqrt{a}$ for all $k \geq 1$. This means that

$$x_k^2 - a \geq 0 \text{ for all } k \geq 2. \tag{2}$$

Next, notice that, for all $k \geq 2$,

$$x_k - x_{k+1} = x_k - \tfrac{1}{2}\left(x_k + \frac{a}{x_k}\right) = \frac{x_k}{2} - \frac{a}{2x_k} = \frac{x_k^2 - a}{2x_k} \geq 0$$

thanks to (2), and the (already established) fact that $x_k > 0$. This means that $x_k \geq x_{k+1}$. This means that

$$x_2 \geq x_3 \geq x_4 \geq \cdots \geq \sqrt{2}.$$

Thus, for $k \geq 2$, x_k is a decreasing sequence which is bounded below by $\sqrt{2}$, so it converges to some point x_*, and also $x_* \geq \sqrt{2} > 0$.

2. Write a Matlab function to compute the limit in Section 5.2 using the sequence. Make the function input the number, `a`, and the number of iterations. Make it output the final iteration, and the absolute value of the error between the final iteration and the exact solution, `sqrt(a)`. Make the code give an error if the input `a` is negative. The whole code will be fairly short, maybe 10 lines or fewer, depending on how you code it. **If you get stuck, review *Part 10: Functions* in the Matlab Introduction file.**

   Solution. Here is one possible example code:

```
function x = sqrtNewton(a,numIter)
% A function to compute the square root of 'a'
% using Newton's method (i.e., the Babylonian method).

if ( a < 0 )
   error('Input must be non-negative');
elseif (a == 0)
   x = 0;
   return;
end

x = a;
for k = 1:numIter
   x = 0.5*(x + a/x);
end

display(sprintf('error = %g',abs(x-sqrt(a))));
```