The Orchestra of Partial Differential Equations

Adam Larios

University of Nebraska Lincoln

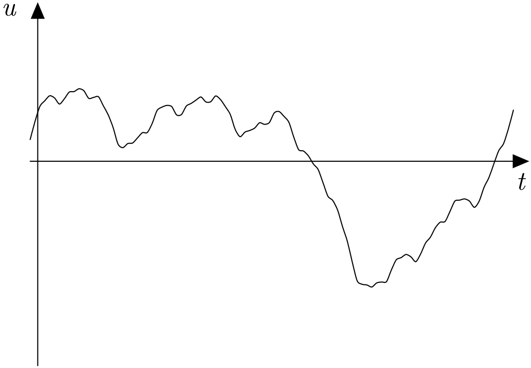
19 January 2017

Landscape Seminar

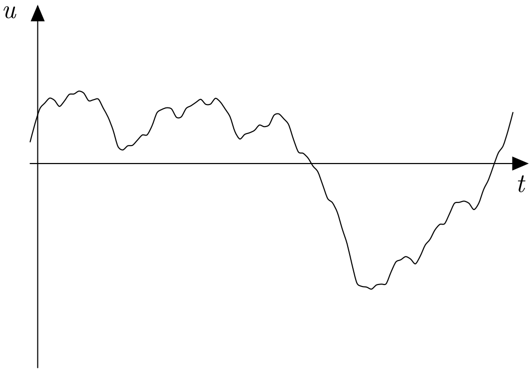
Outline

Outline

Frequency

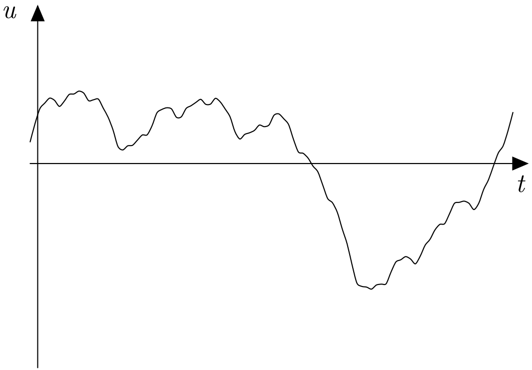


Frequency



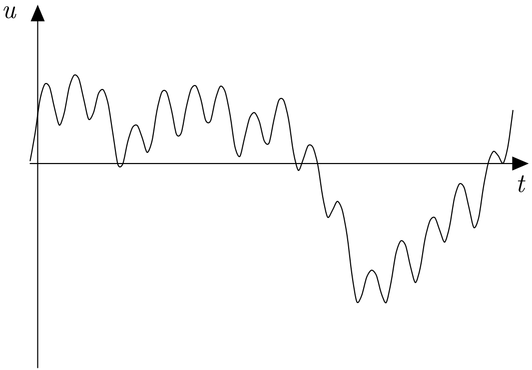
$$u(t) = 0.5\cos(2t) + 0.125\cos(8t) + 0.03125\cos(32t)$$
$$+1.0\sin(1t) + 0.25\sin(4t) + 0.0625\sin(16t)$$

Frequency



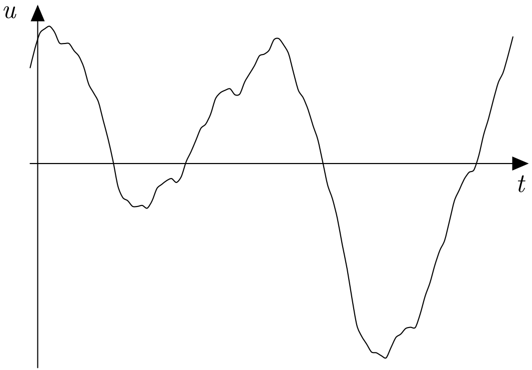
$$u(t) = 0.5\cos(2t) + 0.125\cos(8t) + 0.03125\cos(32t)$$
$$+1.0\sin(1t) + 0.25\sin(4t) + 0.0625\sin(16t)$$

Frequency



$$u(t) = 0.5\cos(2t) + 0.125\cos(8t) + 0.03125\cos(32t)$$
$$+1.0\sin(1t) + 0.25\sin(4t) + 0.3125\sin(16t)$$

Frequency



$$u(t) = 1.5\cos(2t) + 0.125\cos(8t) + 0.03125\cos(32t)$$
$$+1.0\sin(1t) + 0.25\sin(4t) + 0.0625\sin(16t)$$

Fourier Series

$$u(t) = \sum_{k=0}^{\infty} (a_k \cos(kt) + b_k \sin(kt))$$

Fourier Series

$$u(t) = \sum_{k=0}^{\infty} \left(a_k \cos(kt) + b_k \sin(kt) \right)$$

$$e^{ikt} = \cos(kt) + i \sin(kt)$$

$$\cos(kt) = \frac{e^{ikt} + e^{-ikt}}{2}$$

$$\sin(kt) = \frac{e^{ikt} - e^{-ikt}}{2i}$$

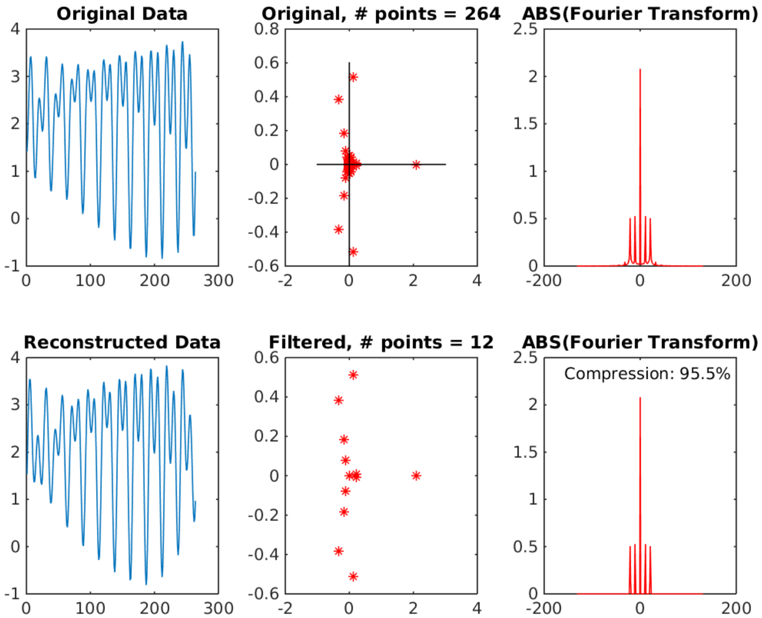
Fourier Series

$$u(t) = \sum_{k=0}^{\infty} \left(a_k \cos(kt) + b_k \sin(kt) \right)$$

$$e^{ikt} = \cos(kt) + i \sin(kt)$$
$$\cos(kt) = \frac{e^{ikt} + e^{-ikt}}{2}$$
$$\sin(kt) = \frac{e^{ikt} - e^{-ikt}}{2i}$$

$$u(t) = \sum_{k=-\infty}^{\infty} c_k e^{ikt}$$

$$c_k = \tfrac{1}{2}\left(a_k - i b_k\right), \quad k > 0,$$
$$c_k = \tfrac{1}{2}\left(a_k + i b_k\right), \quad k < 0.$$


Original Data
Original, # points = 264
ABS(Fourier Transform)
Reconstructed Data
Filtered, # points = 12
ABS(Fourier Transform)
Compression: 95.5%

Multi-dimensional Fourier series

$$u(x) = \sum_{k \in \mathbb{Z}} \widehat{u}_k e^{ikx}$$

$$u(\vec{x}) = \sum_{\vec{k} \in \mathbb{Z}^n} \widehat{u}_{\vec{k}} e^{i\vec{k}\cdot\vec{x}}$$

Multi-dimensional Fourier series

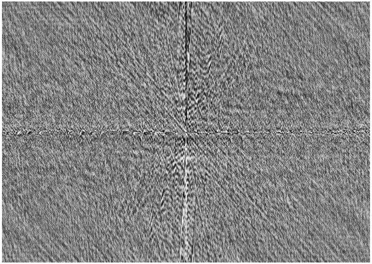
$$u(x) = \sum_{k\in\mathbb{Z}} \widehat{u}_k e^{ikx}$$

$$u(\vec{x}) = \sum_{\vec{k}\in\mathbb{Z}^n} \widehat{u}_{\vec{k}} e^{i\vec{k}\cdot\vec{x}}$$

$$\widehat{u}_{\vec{k}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(\vec{x})\,e^{-i\vec{k}\cdot\vec{x}}\,d\vec{x}$$

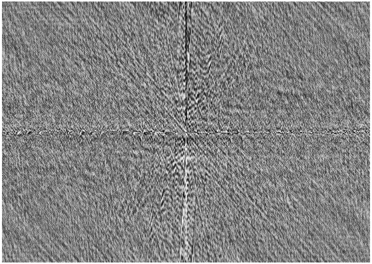
(a) Original image

(a) Original image

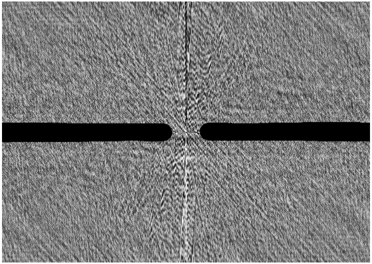


(b) Fourier transform (magnitude)

(a) Original image

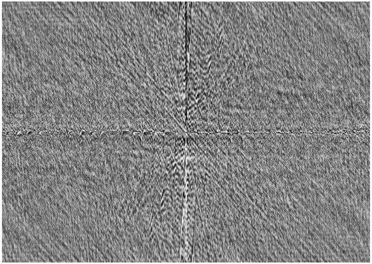


(b) Fourier transform (magnitude)

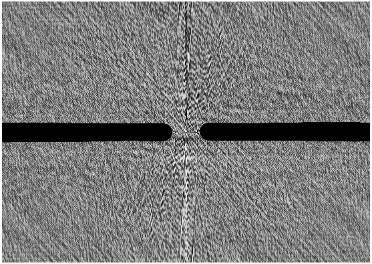


(c) Zero-out Fourier coeffcients

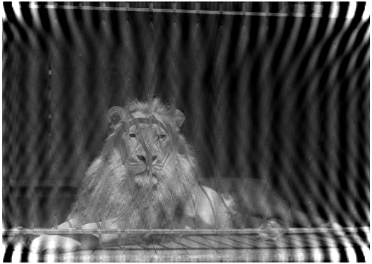
(a) Original image



(b) Fourier transform (magnitude)



(c) Zero-out Fourier coeffcients



(d) Inverse Fourier transform

Derivatives

$$u(x) = \sum_{k\in\mathbb{Z}} \widehat{u}_k e^{ikx}$$

Derivatives

$$u(x) = \sum_{k \in \mathbb{Z}} \widehat{u}_k e^{ikx}$$

$$\frac{d}{dx} u(x) = \sum_{k \in \mathbb{Z}} {\color{red} ik} \widehat{u}_k e^{ikx}$$

Derivatives

$$u(x) = \sum_{k\in\mathbb{Z}} \widehat{u}_k e^{ikx}$$

$$\frac{d}{dx} u(x) = \sum_{k\in\mathbb{Z}} ik\widehat{u}_k e^{ikx}$$

$$\frac{d^2}{dx^2} u(x) = \sum_{k\in\mathbb{Z}} (-k^2)\widehat{u}_k e^{ikx}$$

Derivatives

$$u(x) = \sum_{k\in\mathbb{Z}} \widehat{u}_k e^{ikx}$$

$$\frac{d}{dx} u(x) = \sum_{k\in\mathbb{Z}} ik\widehat{u}_k e^{ikx}$$

$$\frac{d^2}{dx^2} u(x) = \sum_{k\in\mathbb{Z}} (-k^2)\widehat{u}_k e^{ikx}$$

Idea

Can the Fourier transform be used to understand differential equations?

Outline

The Simplest Partial Differential Equation

$$\frac{d\rho}{dt} =$$

$$\text{position} = x = x(t)$$

$$\text{velocity} = v = v(t, x(t)) = \frac{dx}{dt}$$

$$\text{density} = \rho = \rho(t, x(t))$$

The Simplest Partial Differential Equation

$$\frac{d\rho}{dt} = 0$$

$$\text{position} = x = x(t)$$

$$\text{velocity} = v = v(t, x(t)) = \frac{dx}{dt}$$

$$\text{density} = \rho = \rho(t, x(t))$$

The Simplest Partial Differential Equation

$$\frac{d\rho}{dt} = 0$$

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \frac{dx}{dt}\frac{\partial \rho}{\partial x}$$

$$= 0$$

$$\text{position} = x = x(t)$$

$$\text{velocity} = v = v(t, x(t)) = \frac{dx}{dt}$$

$$\text{density} = \rho = \rho(t, x(t))$$

The Simplest Partial Differential Equation

$$\frac{d\rho}{dt} = 0$$

$$\begin{aligned}\frac{d\rho}{dt} &= \frac{\partial \rho}{\partial t} + \frac{dx}{dt}\frac{\partial \rho}{\partial x} \\ &= \frac{\partial \rho}{\partial t} + \quad v\frac{\partial \rho}{\partial x} = 0\end{aligned}$$

$$\begin{aligned}\text{position} &= x = x(t) \\ \text{velocity} &= v = v(t, x(t)) = \frac{dx}{dt} \\ \text{density} &= \rho = \rho(t, x(t))\end{aligned}$$

The Simplest Partial Differential Equation

$$\frac{d\rho}{dt} = 0$$

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \frac{dx}{dt}\frac{\partial \rho}{\partial x} = \frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} = 0$$

Transport Equation

$$\rho_t + v\rho_x = 0$$

$$\text{position} = x = x(t)$$

$$\text{velocity} = v = v(t, x(t)) = \frac{dx}{dt}$$

$$\text{density} = \rho = \rho(t, x(t))$$

The Simplest Partial Differential Equation

$$\frac{d\rho}{dt} = 0$$

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \frac{dx}{dt}\frac{\partial \rho}{\partial x}$$
$$= \frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} = 0$$

Transport Equation

$$\rho_t + v\rho_x = 0$$

Transport Equation in $\mathbb{R}^n$

$$\rho_t + (\vec{v} \cdot \nabla)\rho = 0$$

position $= x = x(t)$

velocity $= v = v(t, x(t)) = \dfrac{dx}{dt}$

density $= \rho = \rho(t, x(t))$

$$(\vec{v} \cdot \nabla)\rho = v_1\frac{\partial \rho}{\partial x} + v_2\frac{\partial \rho}{\partial y} + v_3\frac{\partial \rho}{\partial z}$$

Idea

What about the water itself?
What if we set $\rho = v = u$?

Idea

What about the water itself?
What if we set $\rho = v = u$?

Burgers' Equation

$$u_t + uu_x = 0$$

Burgers' Equation in $\mathbb{R}^n$

$$\vec{u}_t + (\vec{u} \cdot \nabla)\vec{u} = 0$$

Computer Time!

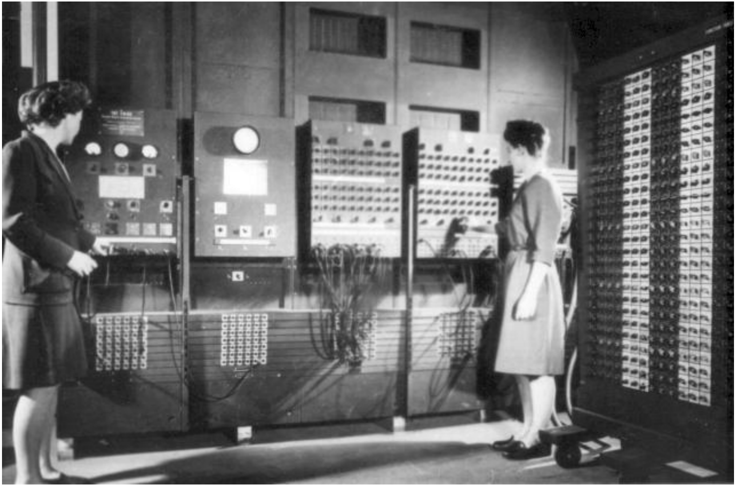


Figure : Programmers working on ENIAC, one of the first computers (c. 1946)

Diffusion Equation

Diffusion

Diffusion Equation

Diffusion

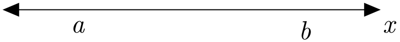
$$\text{concentration} = \theta = \theta(x,t)$$

$$\text{flux} = f = f(x,t) = f(x)$$

Diffusion Equation

Diffusion

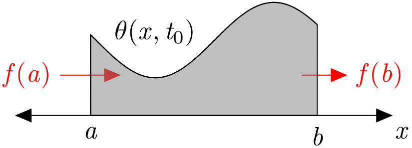
$$\text{concentration} = \theta = \theta(x,t)$$
$$\text{flux} = f = f(x,t) = f(x)$$



Diffusion Equation

Diffusion

$$\text{concentration} = \theta = \theta(x,t)$$
$$\text{flux} = f = f(x,t) = f(x)$$

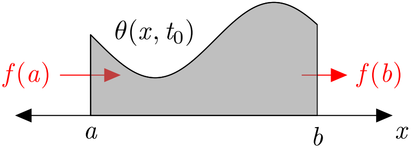


Diffusion Equation

Diffusion

concentration $= \theta = \theta(x,t)$

flux $= f = f(x,t) = f(x)$



$$\frac{d}{dt}\int_a^b \theta(x,t)\,dx = f(a) - f(b)$$

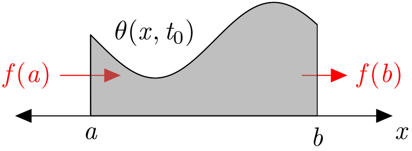
$$= -\int_a^b \frac{\partial f}{\partial x}\,dx$$

Diffusion Equation

Diffusion

concentration $= \theta = \theta(x,t)$

flux $= f = f(x,t) = f(x)$



$$\frac{d}{dt}\int_a^b \theta(x,t)\,dx = f(a) - f(b)$$

$$= -\int_a^b \frac{\partial f}{\partial x}\,dx$$

$$\boxed{\int_a^b \frac{\partial \theta}{\partial t}\,dx = -\int_a^b \frac{\partial f}{\partial x}\,dx}$$

Diffusion Equation

$$\boxed{\int_a^b \frac{\partial \theta}{\partial t}\,dx = -\int_a^b \frac{\partial f}{\partial x}\,dx}$$

Fourier's law:

$$f = -\nu \frac{\partial \theta}{\partial x}, \qquad \nu > 0$$

Diffusion Equation

$$\boxed{\int_a^b \frac{\partial \theta}{\partial t}\,dx = -\int_a^b \frac{\partial f}{\partial x}\,dx}$$

Fourier's law:

$$f = -\nu\frac{\partial \theta}{\partial x}, \qquad \nu > 0$$

$$\int_a^b \frac{\partial \theta}{\partial t}\,dx = \int_a^b \nu\frac{\partial^2 \theta}{\partial x^2}\,dx$$

Diffusion Equation

$$\boxed{\int_a^b \frac{\partial \theta}{\partial t}\,dx = -\int_a^b \frac{\partial f}{\partial x}\,dx}$$

Fourier's law:

$$f = -\nu \frac{\partial \theta}{\partial x}, \qquad \nu > 0$$

$$\int_a^b \frac{\partial \theta}{\partial t}\,dx = \int_a^b \nu \frac{\partial^2 \theta}{\partial x^2}\,dx$$

Diffusion Equation

$$\theta_t = \nu \theta_{xx}$$

Diffusion Equation in $\mathbb{R}^3$

$$\theta_t = \nu(\theta_{xx} + \theta_{yy} + \theta_{zz}) = \nu \triangle \theta$$

Diffusion Equation and the Fourier Transform

$$\boxed{u_t = \nu u_{xx}}$$

$$u(x,t) = \sum_{k\in\mathbb{Z}} \widehat{u}_k(t) e^{ikx}$$

Diffusion Equation and the Fourier Transform

$$\boxed{u_t = \nu u_{xx}}$$

$$u(x,t) = \sum_{k\in\mathbb{Z}} \widehat{u}_k(t)\, e^{ikx}$$

$$u_{xx} = \sum_{k\in\mathbb{Z}} (-k^2)\widehat{u}_k\, e^{ikx}$$

Diffusion Equation and the Fourier Transform

$$\boxed{u_t = \nu u_{xx}}$$

$$u(x,t) = \sum_{k\in\mathbb{Z}} \widehat{u}_k(t)\, e^{ikx}$$

$$u_{xx} = \sum_{k\in\mathbb{Z}} (-k^2)\widehat{u}_k\, e^{ikx}$$

$$u_t = \sum_{k\in\mathbb{Z}} \frac{d}{dt}\widehat{u}_k\, e^{ikx}$$

Diffusion Equation and the Fourier Transform

$$\boxed{u_t = \nu u_{xx}}$$

$$u(x,t) = \sum_{k\in\mathbb{Z}} \widehat{u}_k(t)\, e^{ikx}$$

$$u_{xx} = \sum_{k\in\mathbb{Z}} (-k^2)\widehat{u}_k\, e^{ikx}$$

$$u_t = \sum_{k\in\mathbb{Z}} \frac{d}{dt}\widehat{u}_k\, e^{ikx}$$

Fourier Coefficients

$$\frac{d}{dt}\widehat{u}_k = -\nu k^2 \widehat{u}_k, \quad k \in \mathbb{Z}$$

Diffusion Equation and the Fourier Transform

$$\boxed{u_t = \nu u_{xx}}$$

$$u(x,t) = \sum_{k\in\mathbb{Z}} \widehat{u}_k(t)\, e^{ikx}$$

$$u_{xx} = \sum_{k\in\mathbb{Z}} (-k^2)\widehat{u}_k\, e^{ikx}$$

$$u_t = \sum_{k\in\mathbb{Z}} \frac{d}{dt}\widehat{u}_k\, e^{ikx}$$

Fourier Coefficients

$$\frac{d}{dt}\widehat{u}_k = -\nu k^2 \widehat{u}_k, \quad k \in \mathbb{Z} \qquad \Rightarrow \qquad \widehat{u}_k(t) = e^{-\nu k^2 t}\widehat{u}_k(0)$$

Diffusion Equation and the Fourier Transform

$$\boxed{u_t = \nu u_{xx}}$$

$$u(x,t) = \sum_{k\in\mathbb{Z}} \widehat{u}_k(t)\, e^{ikx}$$

$$u_{xx} = \sum_{k\in\mathbb{Z}} (-k^2)\widehat{u}_k\, e^{ikx}$$

$$u_t = \sum_{k\in\mathbb{Z}} \frac{d}{dt}\widehat{u}_k\, e^{ikx}$$

Fourier Coefficients

$$\frac{d}{dt}\widehat{u}_k = -\nu k^2 \widehat{u}_k, \quad k \in \mathbb{Z} \quad \Rightarrow \quad \widehat{u}_k(t) = e^{-\nu k^2 t}\widehat{u}_k(0)$$

$$u(x,t) = \sum_{k\in\mathbb{Z}} e^{-\nu k^2 t}\widehat{u}_k(0)\, e^{ikx}$$

Computer Time Again!

Figure : Woman working on a Cray supercomputer. (c. 1986)

Backward Diffusion

$$u_t = -\nu u_{xx}$$

Backward Diffusion

$$\boxed{u_t = -\nu u_{xx}}$$

$$u(x,t) = \sum_{k\in\mathbb{Z}} e^{+\nu k^2 t}\widehat{u}_k(0)e^{ikx}$$

Backward Diffusion

$$\boxed{u_t = -\nu u_{xx}}$$

$$u(x,t) = \sum_{k\in\mathbb{Z}} e^{+\nu k^2 t}\widehat{u}_k(0)e^{ikx}$$

Massively unstable!

Fourth-Order Diffusion

$$\boxed{u_t = -\nu u_{xxxx}}$$

Fourth-Order Diffusion

$$\boxed{u_t = -\nu u_{xxxx}}$$

$$u_{xxxx} = \sum_{k\in\mathbb{Z}} (k^4)\widehat{u}_k e^{ikx}$$

Fourth-Order Diffusion

$$\boxed{u_t = -\nu u_{xxxx}}$$

$$u_{xxxx} = \sum_{k\in\mathbb{Z}} (k^4)\widehat{u}_k e^{ikx}$$

$$u(x,t) = \sum_{k\in\mathbb{Z}} e^{-\nu k^4 t}\widehat{u}_k(0)e^{ikx}$$

Massively stable!

Third-Order Dispersion

$$\boxed{u_t = \nu u_{xxx}}$$

Third-Order Dispersion

$$\boxed{u_t = \nu u_{xxx}}$$

$$u_{xx} = \sum_{k\in\mathbb{Z}} (ik^3)\widehat{u}_k e^{ikx}$$

Third-Order Dispersion

$$\boxed{u_t = \nu u_{xxx}}$$

$$u_{xx} = \sum_{k\in\mathbb{Z}} (ik^3)\widehat{u}_k e^{ikx}$$

$$u(x,t) = \sum_{k\in\mathbb{Z}} e^{i\nu k^3 t}\widehat{u}_k(0)e^{ikx}$$

Transport-Diffusion Equation

Transport-Diffusion Equation

$$\rho_t + u\rho_x = \nu\rho_{xx}$$

Transport-Diffusion Equation in $\mathbb{R}^n$

$$\rho_t + (\vec{u}\cdot\nabla)\rho = \nu\triangle\rho$$

Outline

Nonlinear Equations

Burgers Equation [Shock Waves, Traffic]

$$u_t + uu_x = \nu u_{xx}$$

Nonlinear Equations

Burgers Equation [Shock Waves, Traffic]

$$u_t + uu_x = \nu u_{xx}$$

Korteweg-de Vries (KdV) Equation [Water Waves]

$$u_t + uu_x = u_{xxx}$$

Kuramoto-Sivashinsky (KS) Equation [Flames]

$$u_t + uu_x = -\lambda u_{xx} - u_{xxxx}$$

Navier-Stokes Equations [Incompressible Fluids]

$$\begin{cases} \vec{u}_t + (\vec{u} \cdot \nabla)\vec{u} = \nu \triangle \vec{u} - \nabla p \\ \nabla \cdot \vec{u} = 0 \end{cases}$$

Nonlinear Equations

Burgers Equation [Shock Waves, Traffic]

$$u_t + uu_x = \nu u_{xx}$$

Korteweg-de Vries (KdV) Equation [Water Waves]

$$u_t + uu_x = u_{xxx}$$

Kuramoto-Sivashinsky (KS) Equation [Flames]

$$u_t + uu_x = -\lambda u_{xx} - u_{xxxx}$$

Navier-Stokes Equations [Incompressible Fluids]

$$\begin{cases} \vec{u}_t + (\vec{u} \cdot \nabla)\vec{u} = \nu \triangle \vec{u} - \nabla p \\ \nabla \cdot \vec{u} = 0 \end{cases}$$

Computer Time Once More!

Figure : Hopper Cray XE6 at NERSC, named after American computer scientist Dr. Grace Hopper, 1906-1992.

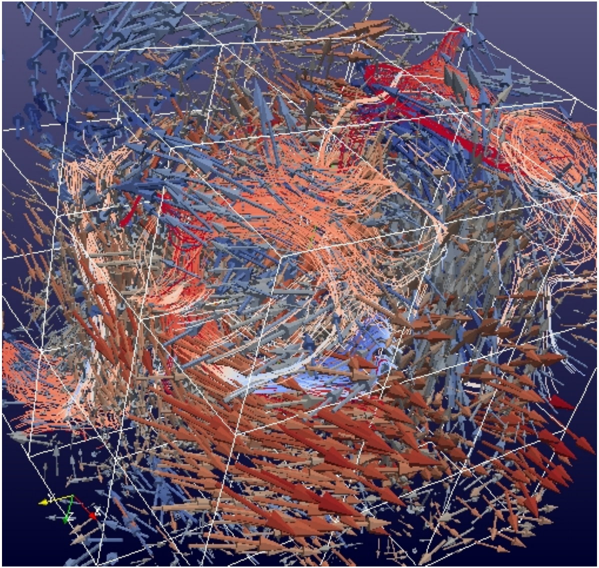


Figure : Simulation of a solution to the 3D Navier-Stokes equations.

The Incompressible Navier-Stokes Equations

Claude L.M.H. Navier

George G. Stokes

Momentum Equation

$$\underbrace{\frac{\partial \vec{u}}{\partial t}}_{Acceleration} + \underbrace{(\vec{u}\cdot\nabla)\vec{u}}_{Advection} = \underbrace{-\nabla p}_{\substack{Pressure\\Gradient}} + \underbrace{\nu\triangle\vec{u}}_{\substack{Viscous\\Diffusion}}$$

Continuity Equation (Divergence-Free Condition)

$$\nabla\cdot\vec{u} = 0$$

Unknowns

$\vec{u}$:= Velocity (vector)

p := Pressure (scalar)

Parameter

ν := Kinematic Viscosity

Problem (Leray 1933)

Existence and uniqueness of strong solutions in 3D for all time. ($1,000,000 Clay Millennium Prize Problem)

Thank you!