Math 308 - Section 506 - Project 4

Instructor: Dr. Adam Larios

Due date: Monday, 9 April 2012

Instructions. While there are no specific neatness requirements, your work should look professional, and you should submit it in a form that would be appropriate for submitting to a boss at a job you care about. Credit will be given for full, complete, and thoughtful analysis; however, a good report is not necessarily a long report. Do not submit fluff or filler, but only quality work that you are proud of.

Submit your project as a group with all group member's names clearly labeled on the front. Submit all of your code[1], along with any relevant graphs, mathematical calculations, and explanations. Make sure any graphs are well labeled and well referenced, and that they are easy to read. Please note that each person is responsible for 100% of the work, so it is your responsibility to keep all group members on task, or to finish the work yourself.

You will need the files `lorenz_demo.m`, `g.m`, and `pplane8.m` from the class webpage. This project involves phase plane analysis. We will investigate the following systems of equations; specifically, the behavior of the systems as the various parameters are changed.

Part I. Two dimensional Analysis.

In the first case, we can use phase plane analysis (nodes, saddle points, periodic orbits, etc). Consider the following system of equations, which is often used to model predator-prey systems, with x representing the population of the predators, and y representing the population of the prey.

The 2×2 Lotka-Volterra (LV) system:

$$\frac{dx}{dt} = x(\alpha - \beta y)$$
$$\frac{dy}{dt} = -y(\epsilon - \delta x)$$

Assuming the parameters α, β, ϵ, and δ are all positive, find the critical point(s) of the LV system in terms of the parameters. Identify the type of critical point(s) by using eigenvalues. Illustrate each type of critical point with a phase plane plot, and component plots, including solution curves.

Extinction occurs if one or more of the species vanishes as $t \to \infty$. Can this happen for solutions of the LV equations? If so, give an example (initial conditions, phase plane plots and component plots). If not, explain why (using mathematical tools and arguments).

[1]There is no need to submit code which was *not* written by your group.

Part II. Three dimensional Analysis.

In this second part, we work in three dimensions, where we will see that things are much more complicated (and interesting!). We can see evidence of a phenomenon called chaos.

The following system was first derived in 1963 by Edward Lorenz, as a simplified mathematical model for atmospheric convection.

The 3×3 Lorenz Dynamical System:

$$\begin{aligned}
&\frac{dx}{dt} = \sigma(y - x),\\
&\frac{dy}{dt} = x(\rho - z) - y,\\
&\frac{dz}{dt} = xy - \beta z,\\
&\text{with } \sigma = 10, \rho = 28, \beta = 83.
\end{aligned}$$

While studying the solutions to this nonlinear equation, Lorenz made a startling accidental discovery: small perturbations of the initial data gave rise to wildly different behavior of the solutions. This was the first time anyone had seen that a dynamical system can give rise to chaotic dynamics. Lorenz's discovery marked the beginning of a new branch of mathematics which studied chaos, and caused a major shift in the way we think about science and predictability. It gave us such beautiful ideas as the notion of "strange attractors" (which are intimately connected with fractals) and also a phenomenon that Lorenz named the "butterfly effect."

Change the parameters σ, ρ, and β from the values given above, and examine the behavior (especially near the critical point). You may change one or more values at a time. Find as many different sets of parameters that give rise to fundamentally different behavior. Describe the behavior in the phase plane, as well as in the component planes $x(t)$ vs t, $y(t)$ vs t, and $z(t)$ vs t. Compare and contract your observations with the typical behavior of linear systems that we have studied before.