A WORKED-OUT EXAMPLE FOR REPEATED ROOTS
MATH 308

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Consider the 2×2 linear system:

$$\dot{\mathbf{x}} = A\mathbf{x} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \mathbf{x}$$

Our goal is to find the generalized solution to this system. As usual, first, we look for the eigenvalues of the matrix. We compute

$$\det(A) = \det \begin{pmatrix} 3-\lambda & -4 \\ 1 & -1-\lambda \end{pmatrix} \mathbf{x} = (3-\lambda)(-1-\lambda) - (-4)(1) = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$$

The eigenvalues are the values that mat this determinant equal to 0, so $\lambda = 1$ is the only eigenvalue. To compute the eigenvector for $\lambda = 1$, that is, a vector [1] $\boldsymbol{\xi} = \left(\begin{smallmatrix} \xi_1 \\ \xi_2 \end{smallmatrix}\right)$ such that $A\boldsymbol{\xi} = \lambda\boldsymbol{\xi} = \boldsymbol{\xi}$, we must find a solution to the system $(A - \lambda I)\boldsymbol{\xi} = \mathbf{0}$. That is, we must solve

$$\begin{aligned} 2\xi_1 - 4\xi_2 &= 0, \\ 1\xi_1 - 2\xi_2 &= 0. \end{aligned}$$

As expected, these equations are redundant (the second is two times the first). Thus, we only need ξ_1 and ξ_2 to satisfy $\xi_1 = 2\xi_2$. Thus we have, for example, $\boldsymbol{\xi} = \left(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right)$ is an eigenvector (and so is any non-zero multiple of $\left(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right)$ of course).

According to what we learned from earlier in Chapter 7, this means that *one* solution is given by

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} 2e^t \\ e^t \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^t = \boldsymbol{\xi}\, e^t.$$

However, we are not done. Remember: we want to find *all* the solutions, but we only have one (repeated) root, so we have to resort to another method. We now look for a second solution which is linearly from the one we found. We look for one of the form (see your notes for motivation for this choice of form):

$$\mathbf{x}^{(2)}(t) = \boldsymbol{\eta}\, e^t + t\,\boldsymbol{\xi}\, e^t$$

Next, we want to substitute this into the original system $\dot{\mathbf{x}} = A\mathbf{x}$. First, we compute:

$$\frac{d\mathbf{x}^{(2)}}{dt} = \boldsymbol{\eta}\, e^t + \boldsymbol{\xi}\, e^t + t\,\boldsymbol{\xi}\, e^t$$

where we have used the product rule on the last term. Next, we compute

$$A\mathbf{x}^{(2)} = A(\boldsymbol{\eta}\, e^t + t\,\boldsymbol{\xi}\, e^t) = A\boldsymbol{\eta}\, e^t + t\, A\boldsymbol{\xi}\, e^t = A\boldsymbol{\eta}\, e^t + t\,\boldsymbol{\xi}\, e^t$$

since $A\boldsymbol{\xi} = \lambda\boldsymbol{\xi} = \boldsymbol{\xi}$. Therefore, the equation $\frac{d\mathbf{x}^{(2)}}{dt} = A\mathbf{x}^{(2)}(t)$ becomes

$$\begin{aligned} & \boldsymbol{\eta}\, e^t + \boldsymbol{\xi}\, e^t + \cancel{t\,\boldsymbol{\xi}\, e^t} = A\boldsymbol{\eta}\, e^t + \cancel{t\,\boldsymbol{\xi}\, e^t} \\ \Rightarrow \quad & \boldsymbol{\eta}\, \cancel{e^t} + \boldsymbol{\xi}\, \cancel{e^t} = A\boldsymbol{\eta}\, \cancel{e^t} \\ \Rightarrow \quad & (A - I)\boldsymbol{\eta} = \boldsymbol{\xi}. \end{aligned}$$

That is, we must solve the redundant system

$$\begin{aligned} 2\eta_1 - 4\eta_2 &= 2, \\ 1\eta_1 - 2\eta_2 &= 1. \end{aligned}$$

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[1] ξ is pronounced like "ka-SEE", and η is pronounced like "EIGHT-uh"

Any solution will do, and we are free to choose one of the variables, so let us pick $\eta_2 = 0$ for simplicity. Then, we must have $\eta_1 = 1$, and our second solution becomes

$$\mathbf{x}^{(2)}(t) = \boldsymbol{\eta}\, e^t + t\, \boldsymbol{\xi}\, e^t = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + te^t \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} e^t + 2te^t \\ te^t \end{pmatrix}.$$

Finally, we have two solutions. The question remains: are these two solutions linearly independent? If they are not, all our work for constructing $\mathbf{x}^{(2)}(t)$ was worthless. Luckily, as it turns out, the method we used above will always produce another linearly independent solution. However, it is worth checking to see if they really are linearly independent. To do this, we check the Wronskian, and see if they are not zero:

$$W[\mathbf{x}^{(1)}(t), \mathbf{x}^{(2)}(t)] = \det \begin{pmatrix} 2e^t & e^t + 2te^t \\ e^t & te^t \end{pmatrix} = (2e^t)(te^t) - (e^t)(e^t + 2te^t) = 2te^{2t} - e^{2t} - 2te^{2t} = -e^{2t} \neq 0$$

Thus, the Wronskian is never zero, so the solutions are linearly independent. That means we are basically done! Now, to form all the solutions, we just need to take linear combinations of $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$. Therefore, the general solution is given by

$$\mathbf{x}(t) = c_1 \mathbf{x}^{(1)}(t) + c_2 \mathbf{x}^{(2)}(t) = c_1 \begin{pmatrix} 2e^t \\ e^t \end{pmatrix} + c_2 \begin{pmatrix} e^t + 2te^t \\ te^t \end{pmatrix}.$$

for arbitrary c_1 and c_2.

Now our work here is done, so it's time to go get some tacos.

WAIT! *Before* we get some tacos, let's plot this in pplane8 really quick! Put the system in and look at how it behaves along the line spanned by the eigenvector (remember that there is only one eigenvector here). What do you see? Is it a source, a sink, a saddle, or something else? Try it out! After you do, maybe *then* it will be time to go get some tacos. When you come back, try an example like this on your own, just for practice. Have fun!