MATH 609-600
Sample Exam #1
Numerical Linear Algebra

The exam will be based on the material covered in the area of numerical linear algebra. You should be able to prove any major result that we had in class.

Also there will be a substantial body of problems that you need to compute certain quantities in order to get an answer to the question. Remember the Taussky's maxim and use it when you can: if a conjecture about a matrices is false, it can usually be disproved with a 2×2 exapmle.

Here is a set of practice problems you can work on in order to prepare yourself for the exam. Do not expect that problems on the exam will be part of these problems, though there might be some similar problems.

Below we shall use the following matrices:

$$W = \begin{bmatrix} 3 & 2 & 0 & 1 \\ 2 & 5 & 1 & 1 \\ 0 & 2 & 3 & 0 \\ 1 & 1 & 0 & 3 \end{bmatrix} \qquad V = \begin{bmatrix} 3 & 2 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \qquad Z = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

1. Some simple and not so simple problems

(1) Prove that the inverse of a nonsingular block upper triangular matrix is a upper block-triangular matrix.

(2) Here $A \in \mathbb{R}^{n\times n}$ is nonsingular and $u, v \in \mathbb{R}^n$.

$$S = \begin{bmatrix} 0 & u^T \\ v & A \end{bmatrix}$$

Under what condition S is invertible, and give a formula for the inverese when it exists.

(3) Let D be a matrix in partitioned form:

$$D = \begin{bmatrix} A & B \\ C & I \end{bmatrix}$$

Prove that if $A - BC$ is nonsingular, then so is D.

(4) Is matrix Z positive definite ? Prove or disprove.

(5) Assume A and B are symmetric matrices in $\mathbb{R}^{n\times n}$. Prove that AB has real eigenvalues.

(6) Find the spectral radius of the matrix V.

(7) For what values of a is the matrix A is not positive definite ?

$$A = \begin{bmatrix} 1 & a \\ a & 1 \end{bmatrix}$$

(8) Is the matrix W invertible ?

(9) Let $\|\cdot\|$ be a subordinate matrix norm, and let S be a nonsingular matrix. Define $\|A\|_* = \|SAS^{-1}\|$. Show that $\|\cdot\|_*$ is a subordinate norm.

(10) Assume that A and B are such that for some subordinate matrix norm $\|\cdot\|$ the following holds true: $\|I - BA\| < 1$. Show that both A and B are nonsingular.

(11) Assume that A and B are such that for some subordinate matrix norm $\|\cdot\|$ the following is true: $\|I - BA\| < 1$. Prove or disprove that $\|I - AB\| < 1$.

(12) Consider the matrix $A \in \mathbb{R}^{2n\times 2n}$ of the block form $A = \begin{bmatrix} A_1 & 0 \\ \text{C} & A_2 \end{bmatrix}$, where C is an arbitrary matrix in $\mathbb{R}^{n\times n}$ and $A_1,\ A_2 \in \mathbb{R}^{n\times n}$ are given by

$$A_1 = \begin{bmatrix} 2 & 1 & 0 & \dots & 0 & 0 \\ 1 & 2 & 1 & \dots & 0 & 0 \\ 0 & 1 & 2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 2 & 1 \\ 0 & 0 & 0 & \dots & 1 & 2 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 2 & \text{-1} & 0 & \dots & 0 & 0 \\ \text{-1} & 2 & \text{-1} & \dots & 0 & 0 \\ 0 & \text{-1} & 2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 2 & \text{-1} \\ 0 & 0 & 0 & \dots & \text{-1} & 2 \end{bmatrix}$$

(1) Show that A is a regular matrix (that means, nonsingular).
(2) Find a formula for its inverse in terms of A_1^{-1}, A_2^{-1} and C. Hint: Realise that the inverse of A is an upper block triangular matrix, so the formula should provide expressions for the corresponding blocks.
(3) If $C = 0$ then A is symmenrtic. Is this matrix positive definite ? (4) Find its spectrum for $n = 2$.

(13) Consider the Jacobi iteration for the system $Zx = b$. Does it converege ? Explain !

(14) Write down the Richardson method for the system $Zx = b$. What is the optimal parameter τ for the Richardson iteration.

(15) Can you show convergence of an iterative method of your choice for the system $Wx = b$.

(16) Show that Jacobi method for the system $A_1x = b$ (A_1 is the matrix defined in Problem 12) is convergent.

(17) Show that Jacobi method for the system $A_2x = b$ (A_2 is the matrix defined in Problem 12) is convergent.

(18) Let U be a real upper tiangular $n \times n$ matrix, i.e. $u_{ij} = 0$ for $i > j$. Answer the following questions:
(a) When this matrix is nonsingular;
(b) find its spectral radius;
(c) if U is nonsingular show that its inverse is also upper triangular.

(19) Let A and B be matrices in $\mathbb{R}^{n\times n}$. Assume that A is nonsingular and A and B satisfy the inequality $\|A^{-1}\|\,\|B\| < 1$ for some subordinate matrix norm. Show that the matrix $A+B$ is nonsingular.

(20) Given the matrices

$$A = \begin{bmatrix} 0 & 1 \\ \text{-2} & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ \text{-1/2} & 0 \end{bmatrix}, \quad C = \begin{bmatrix} \text{-2} & 1 \\ 0 & 2 \end{bmatrix}.$$

(a) Compute $\|A\|_2$ and $\|C\|_2$.
(b) Does the Jacobi iterative method for $Cx = b$ converge ?
(c) Does the iteration $x^{(k+1)} = B\,x^{(k)} + c$ converge for any intial guess $x^{(0)}$?

(21) Consider the linear system $Ax = b$, $x \in \mathbb{R}^n$ with A an SPD matrix. Let λ_n and λ_1 be the largest and the smallest eigenvalues of A. Recall, that Richardson method

$$x^{(k+1)} = (I - \tau A)x^{(k)} + \tau b, \quad \text{with optimal parameter} \quad \tau = \frac{2}{\lambda_n + \lambda_1}$$

satisfies the following error transfer estimate (here $e^{(k)} = x^{(k)} - x$):

$$\|e^{(k+1)}\|_2 \le \frac{\lambda_n - \lambda_1}{\lambda_n + \lambda_1}\|e^{(k)}\|_2, \quad \text{where } \|\cdot\|_2 \text{ is the Euclidean norm.}$$

Using the eigenvectors (and eigenvalues) of A (this is called Fourier mode analysis) prove that the following estimate is also valid

$$\|e^{(k+1)}\|_A \le \frac{\lambda_n - \lambda_1}{\lambda_n + \lambda_1}\|e^{(k)}\|_A, \quad \text{where } \|x\|_A = (Ax, x)^{\frac{1}{2}}.$$

(22) Consider the linear system $Ax = b$, $x \in \mathbb{R}^n$ with A nonsingular, but NOT positive (neither negative) definite matrix. This means that A has positive and negative eigenvalues. Show that there is NO parameter τ so that the Richardson iteration

$$x^{(k+1)} = (I - \tau A)x^{(k)} + \tau b$$

converges.

(23) Consider the linear system $Ax = b$, $x \in \mathbb{R}^n$ with A an SPD matrix. Assume that the error $e^{(m)} = x^{(m)} - x$ of an iterative method satisfies the estimate

$$\|e^{(m)}\|_A \le \|Q_m(A)\|_A \, \|e^{(0)}\|_A,$$

where $Q_m(A)$ is a polynomial of degree which satrisfies $Q_m(0) = I$. Show that

$$\|Q_m(A)\|_A = \max_{i=1,\dots,n} |Q_m(\lambda_i)|.$$

where λ_i are the eigenvalues of A.

Remark. This problem is a generalization of Problem 21, but the proof is essentially the same and relies on the representation of any vector in $\mathbb{R}^n$ by the eigenvectors of A.

(24) Show that the norm of the plynomial Q_m from Problem 23 satisfies the following estimate

$$\max_{i=1,\dots,n} |Q_m(\lambda_i)| \le 2\Big(\frac{1-\sqrt{\xi}}{1+\sqrt{\xi}}\Big)^m, \quad \text{where} \quad \xi = \frac{\lambda_1}{\lambda_n}.$$

2. Solutions

Problem 1

Prove that the inverse of a nonsingular block upper triangular matrix is a upper block-triangular matrix.

Solution: The general form of block upper triangular matrix is

$$U = \begin{bmatrix} A & O \\ C & B \end{bmatrix}$$

where A and B are nonsingular square matrices. The inverse of U in general will have the following block form

$$U^{-1} = \begin{bmatrix} X & Y \\ W & Z \end{bmatrix}$$

where the the unknown blocks shoud be chosen in such way that $UU^{-1} = I$. Performing the block multiplications we get

$$AX = I, \quad AY = 0, \quad AW = C, \quad CX + BW = 0, \quad CY + BZ = I.$$

Since A is nonsingular, from $AY = 0$ we get $Y = 0$ as a matrix and, therefore, U^{-1} is an upper block triangular matrix.

Note that from these equations we can determine the other unknown blocks:

$$X = A^{-1}, \quad W = -B^{-1}CA^{-1}, \quad Z = B^{-1}$$

so that we get the following formula for U^{-1}:

$$U^{-1} = \begin{bmatrix} A^{-1} & 0 \\ -B^{-1}CA^{-1} & B^{-1} \end{bmatrix}.$$

Problem 2
Here $A \in \mathbb{R}^{n\times n}$ is nonsingular and $u, v \in \mathbb{R}^n$.

$$S = \begin{bmatrix} 0 & u^T \\ v & A \end{bmatrix}$$

Under what condition S is invertible, and give a formula for the inverese when it exists.

This problem is quite similar to one of the probelms from your HW#1:

Problem 3
Let D be a matrix in partitioned form:

$$D = \begin{bmatrix} A & B \\ C & I \end{bmatrix}$$

Prove that if $A - BC$ is nonsingular, then so is D.

This problem is quite similar to one of the probelms from your HW#1:

Problem 4
Let $\|\cdot\|$ be a subordinate matrix norm, and let S be a nonsingular matrix. Define $\|A\|_* = \|SAS^{-1}\|$. Show that $\|\cdot\|_*$ is a subordinate norm.

One possible solution: First, using the fact that $\|\cdot\|$ is a subordinate matrix norm, we show that $\|\cdot\|_*$ is a norm:

(1) $\|A\|_* = \|SAS^{-1}\| \geq 0$. S is nonsingular, $\|A\|_* = \|SAS^{-1}\| = 0$ if and only if A is a zero matrix.
(2) $\|\alpha A\|_* = \|\alpha SAS^{-1}\| = \alpha\|SAS^{-1}\| = \alpha\|A\|_*$, for any scalar $\alpha \in C$.
(3) $\|A+B\|_* = \|S(A+B)S^{-1}\| = \|SAS^{-1}+SBS^{-1}\| \leq \|SAS^{-1}\|+\|SBS^{-1}\| = \|A\|_*+\|B\|_*$.

Now prove that $\cdot\|_*$ is a subordinate matrix norm.

$$\begin{aligned} \|A\|_* &= \|SAS^{-1}\| \\ &= \max_{x\in C^n,\, x\neq 0} \frac{\|SAS^{-1}x\|}{\|x\|} \\ &= \max_{y\in C^n,\, y\neq 0} \frac{\|SAy\|}{\|Sy\|} \qquad (y = S^{-1}x) \end{aligned}$$

Define a vector norm $\|x\|_* = \|Sx\|$, then

$$\|A\|_* = \max_{y\in C^n,\, y\neq 0} \frac{\|Ay\|_*}{\|y\|_*}$$

so $\|A\|_*$ is the matrix norm induced by the vector norm $\|x\|_*$.

Problem 5

This is similar to one of your HW problems:

Problem 6

This is similar to one of your HW problems:

Problem 7

This is similar to one of your HW problems:

Problem 8

Solution:
There are many ways to address this question. (1) Reduce the matrix to lower triangula matris via elementary transformations; (2) compute the determinats; (3) solve the homogeneous sytem $Wx = 0$.
The easiest of these is to compute the determinant

$$det(W) = W = 3\det\begin{bmatrix} 5 & 1 & 1 \\ 2 & 3 & 0 \\ 1 & 0 & 3 \end{bmatrix} - 2\det\begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 3 \end{bmatrix} - \det\begin{bmatrix} 2 & 5 & 1 \\ 0 & 2 & 3 \\ 1 & 1 & 0 \end{bmatrix} = 71$$

Thus, the matrix is nonsingular and therefore invertible.

Problem 9

This is one of your HW problems:

Problem 10

Solution:
Assume that A is singular. Then there is a nonzero vector ψ such that $A\psi = 0$. then

$$\|I - BA\| = \sup_{x\in\mathbb{R}^n} \frac{\|(I - BA)x\|}{\|x\|} \geq \frac{\|(I - BA)\psi\|}{\|\psi\|} = \frac{\|\psi\|}{\|\psi\|} = 1.$$

However, this contradict the assumption that $\|I - BA\| < 1$. Therefoore, A cannot be singular. The same way we show that B cannot be singular.

Problem 11

Solution:
Use the Taussky's maxim to disprove the statement with a construction of 2×2 exapmle.

Problem 12

Solution:

(1) A is a block-triangular matrix, so to show that it is nonsingular it is sufficient to show that the diagonal blocks are nonsingular matrices. Thus we need to check that A_1 and A_2 are nonsingular. This is one of the HW problems, so you repeat the arguments from there.

(2) The solution has been provided in Problem $\#$ 1 of this set of problems.

(3) Since the diagonal blocks are symmetric, then the matrix (with $C = 0$) is symmetric. The positive definiteness forllows for the fact that both diagonal blocks are positive definite, but you HAVE to show this (this is one of the HW problems).