MATH609-600

Homework #1 Solutions

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1. Problems

This contains a set of possible solutions to all problems of HW-1 and an additional practice problems. Be vigilant since typos are possible (and inevitable).

(1) (Problem 1a, p. 158 in the textbook) Prove that the inverse of a nonsingular upper-triangular matrix is also upper-triangular.

(2) (Problem 12, p. 148 in the textbook) Let A be an $n \times n$ matrix, and let u and v be two vectors in $\mathbb{R}^n$. Consider the $(n+1) \times (n+1)$ matrix M given by

$$M := \begin{pmatrix} A & u \\ v^T & 0 \end{pmatrix}$$

   (a) Find necessary and sufficient conditions on u and v that make the following matrix invertible.
   (b) Give a formula for M^{-1} when it exists.

(3) (Problem 8b, p. 159 in the textbook) Write the **column version** of the Doolittle algorithm, which computed the k^{th} column of L and the k^{th} column of U at the k^{th} step. (Consequently, the order of computing is $u_{1k},\ u_{2k},\ \ldots,\ u_{kk},\ \ell_{k+1,k},\ \ldots,\ \ell_n k$ at the k^{th} step.) Count the number of arithmetic operations (counting the additions/subtractions and multiplications/divisions separately).

(4) (Problem 26, p. 160 in the textbook) Prove: A is positive definite and B is nonsingular if and only if BAB^T is positive definite.

(5) (Problem 27, p. 160 in the textbook) If A is positive definite and invertible, does it follow that A^{-1} is also positive definite? Prove or give a counter-example.

(6) (Problem 16, p. 148 in the textbook) For what values of a is the following matrix positive definite?

$$A = \begin{pmatrix} 1 & a & a \\ a & 1 & a \\ a & a & 1 \end{pmatrix}$$

(7) Discuss the positive-definiteness of the following the matrices.

$$A = \begin{pmatrix} 3 & 2 & 0 & 1 \\ 2 & 5 & 2 & 1 \\ 0 & 2 & -1 & 0 \\ 1 & 1 & 0 & 3 \end{pmatrix} B = \begin{pmatrix} 3 & 2 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} C = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

2. Solutions

(1) Problem 1: Proof: Let $U \in R^{n\times n}$ be a nonsingular upper triangular matrix. Since it is nonsingular, so there exist a matrix $A \in R^{n\times n}$, s.t., $AU = I$, where I is the identity matrix in $R^{n\times n}$.

Let e_i be the unit vector in R^n with 1 on its ith component, for $i = 1, \cdots, n$, and let $A = [a_1, a_2, \cdots, a_n]$, where $\{a_i\}$, $i = 1, \cdots, n$ are the column vectors of A. Then from $AU = I$ we get the following linear system:

$$\begin{cases} a_1 u_{11} = e_1 \\ a_1 u_{12} + a_2 u_{22} = e_2 \\ \cdots \\ a_1 u_{1j} + a_2 u_{2j} + \cdots + a_j u_{jj} = e_j \\ \cdots \\ a_1 u_{1n} + a_2 u_{2n} + \cdots + a_n u_{nn} = e_n \end{cases} \tag{1}$$

solve the linear system above, we have,

$$a_1 = e_1/u_{11}$$

$$a_2 = (e_2 - a_1 u_{12})/u_{22}$$

$$\cdots$$

$$a_j = (e_j - \textstyle\sum_{k=1}^{j-1} a_k u_{kj})/u_{jj}, \quad \text{for } j = 1, \cdots, n.$$

So, we can see that for each column vector a_j, $j = 1, \cdots, n$, it has zeros on the components that have indexes large than j, therefore, the matrix $A = [a_1, \cdots, a_n]$ is an upper triangular matrix.

(2) Problem 2: The necessary and sufficient condition for this matrix to be invertible is that its rank is $n + 1$. Using Block Gaussian Elimination we get:

$$\begin{bmatrix} A & u \\ 0 & -v^T A^{-1} u \end{bmatrix}$$

Because A invertible, i.e. the rank of $[A\ u]$ is n, so the rank of the matrix above is $n + 1$ if and only if $v^T A^{-1} u \neq 0$.

Let

$$\begin{bmatrix} A & u \\ v^T & 0 \end{bmatrix} \begin{bmatrix} B & d \\ e^T & c \end{bmatrix} = I_{n+1} \tag{2}$$

multiplying $[I_n\ 0; -v^T A^{-1}\ 1]$ on both sides, we get

$$\begin{bmatrix} A & u \\ 0 & -v^T A^{-1} u \end{bmatrix} \begin{bmatrix} B & d \\ e^T & c \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ -v^T A^{-1} & 1 \end{bmatrix} \tag{3}$$

$(2,2)$ block of eq(3) gives,

$$(-v^TA^{-1}u)c = 1 \longrightarrow c = -\frac{1}{v^TA^{-1}u}$$

$(2,1)$ block of eq(3) gives,

$$(-v^TA^{-1}u)e^T = -v^TA^{-1} \longrightarrow e = \frac{A^{-T}v}{v^TA^{-1}u}$$

$(1,2)$ block of eq(2) gives,

$$Ad + cu = 0 \longrightarrow d = \frac{A^{-1}u}{v^TA^{-1}u}$$

$(1,1)$ block of eq(2) gives,

$$AB + ue^T = I_n \longrightarrow B = A^{-1} - \frac{(A^{-1}u)(v^TA^{-1})}{v^TA^{-1}u}$$

i.e. the inverse matrix is

$$\begin{bmatrix} A^{-1} - \dfrac{(A^{-1}u)(v^TA^{-1})}{v^TA^{-1}u} & \dfrac{A^{-1}u}{v^TA^{-1}u} \\ \dfrac{v^TA^{-1}}{v^TA^{-1}u} & -\dfrac{1}{v^TA^{-1}u} \end{bmatrix}.$$

(3) <u>Problem 3:</u> Solution:

$$\begin{aligned} A &= LU \\ &= \begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 \\ l_{2,1} & 1 & \cdots & 0 & \cdots & 0 \\ & & \ddots & 0 & 0 & \\ l_{k,1} & & \cdots & 1 & \vdots & 0 \\ l_{k+1,1} & & \cdots & \mathbf{l_{k+1,k}} & 0 & \\ & & & & \ddots & 0 \\ l_{n,1} & & \cdots & \mathbf{l_{n,k}} & \cdots & 1 \end{bmatrix} \begin{bmatrix} u_{1,1} & u_{1,2} & \cdots & \mathbf{u_{1,k}} & \cdots & & u_{1,n} \\ 0 & u_{2,2} & \cdots & \mathbf{u_{2,k}} & \cdots & & u_{2,n} \\ 0 & 0 & \ddots & \mathbf{u_{k-1,k}} & \cdots & & u_{k-1,n} \\ 0 & 0 & \cdots & \mathbf{u_{k,k}} & \cdots & & u_{k,n} \\ 0 & 0 & & & \ddots & & \\ 0 & 0 & & & & u_{n-1,n-1} & u_{n-1,n} \\ 0 & 0 & \cdots & & \cdots & 0 & u_{n-1,n} \end{bmatrix} \end{aligned}$$

The Doolittle algorithm computes the k-th column of L and U at k-th step:

$j \leq k$:

for $k = 1, \cdots, n,$

for $j = 1, \cdots k$

$$u_{j,k} = a_{j,k} - \sum_{s=1}^{j-1} l_{j,s}u_{s,k}$$

$j > k$:

for $k = 1, \cdots, n-1,$

for $j = k+1, \cdots, n$

$$l_{j,k} = (a_{j,k} - \sum_{s=1}^{k-1} l_{j,s}u_{s,k})/u_{k,k}$$

Arithmetic operation counts:

additions/subtractions:

$$
\begin{aligned}
& \sum_{k=1}^{n}\sum_{j=1}^{k}(j-1)+\sum_{k=1}^{n-1}\sum_{j=k+1}^{n}(k-1) \\
= \; & n(n+1)(2n+1)/12-n(n+1)/4+(n-1)^2(n-2)/2-(n-2)(n-1)(2n-3)/6 \\
= \; & n^3/3+O(n^2)
\end{aligned}
$$

multiplications/divisions:

$$
\begin{aligned}
& \sum_{k=1}^{n}\sum_{j=1}^{k}(j-1)+\sum_{k=1}^{n-1}\sum_{j=k+1}^{n}k \\
= \; & n(n+1)(2n+1)/12-n(n+1)/4+n^2(n-1)/2-(n-1)n(2n-1)/6 \\
= \; & n^3/3+O(n^2)
\end{aligned}
$$

(4) Problem 4: Proof: $\Longleftarrow$ A is positive definite and B is nonsingular. $\forall x \neq 0$, we have

$$x^TBAB^Tx = y^TAy > 0, \quad \text{where } y = B^Tx \neq 0$$

so, BAB^T is positive definite.

$\Longrightarrow$ BAB^T is positive definite. First prove that B is nonsingular. Suppose B is singular, then $\exists\, z \neq 0$ s.t. $B^Tz = 0$, then $BAB^Tz = 0$, it is contrary to the positive definiteness of BAB^T, so B is nonsingular. $\forall x \neq 0$, we have

$$x^TAx = x^TB^{-1}BAB^TB^{-T}x = y^TBAB^Ty > 0, \quad \text{where } y = B^{-T}x \neq 0$$

so, A is positive definite.

(5) Problem 5: Solution: Yes, if A is positive definite then A^{-1} is also positive definite. Proof is as following: $\forall x \neq 0$,

$$x^TA^{-1}x = x^TA^{-1}A^TA^{-T}x = y^TA^Ty = y^TAy > 0, \quad \text{where } y = A^{-T}x \neq 0.$$

(6) Problem 6: **A solution:** For a symmetric and positive definite metrix the determinants of all principle minors are positive. First, obviously $A_1 = 1 > 0$. Next,

$$A_2 \; = det\begin{bmatrix} 1 & a \\ a & 1 \end{bmatrix} = 1-a^2 > 0 \quad \Longrightarrow -1 < a < 1$$

and

$$A_3 \; = \; det\begin{bmatrix} 1 & a & a \\ a & 1 & a \\ a & a & 1 \end{bmatrix} = 1-3a^2+2a^3 > 0 \quad \Longrightarrow -1/2 < a.$$

Therefore, the intersection of the intervals $-1 < a < 1$ and $-1/2 < a$ will give $-1/2 < a < 1$.

A complete solution: Solution: A is symmetric, so there exist an orthogonal matrix V s.t. $AV = V\Lambda$, where Λ is a diagonal matrix, with A's eigenvalues on the diagonal, and V's columns are the corresponding eigenvectors. So, $\forall x \neq 0$, we have $x^TAx = x^TV\Lambda V^Tx = y^T\Lambda y$, where $y = V^Tx \neq 0$. Therefore, if all eigenvalues of A is positive, then we have $x^TAx = y^T\Lambda y > 0$.

Now we obtain the eigenvalues of the matrix A. Let λ be an eigenvalue and $x = [x_1, x_2, x_3]^T$ be the corresponding eigenvector:

$$\begin{bmatrix} 1-\lambda & a & a \\ a & 1-\lambda & a \\ a & a & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

the characteristic equation is

$$\begin{aligned} 0 &= det \begin{bmatrix} 1-\lambda & a & a \\ a & 1-\lambda & a \\ a & a & 1-\lambda \end{bmatrix} \\ &= (1-\lambda)((1-\lambda)^2 - a^2) - a(a(1-\lambda) - a^2) + a(a^2 - a(1-\lambda)) \\ &= ((1-\lambda) - a)((1-\lambda)(1-\lambda+a) - 2a^2) \\ &= ((1-\lambda) - a)^2(1-\lambda+2a) \end{aligned}$$

therefore, the eigenvalues are

$$\lambda = 1-a, 1+2a$$

For A to be positive definite requires

$$1-a > 0, 1+2a > 0$$

i.e.

$$-1/2 < a < 1.$$

(7) Problem 7: Solution: Matrix A is obviously NOT and SPD matrix since $a_{33} = -1$. Recall, an SPD matrix should have all diagonal elements positive since taking vector $e_i = (0, \ldots, 1, \ldots 0)^T$ (all entries are zero except i-th element) we get $0 < e_i^T A e_i = a_{ii}$.

Matrix B consists of 2 2×2 diagonal blocks. In order B to be positive both blocks need to be positive. The first block is while the second block is NOT positive definite. Equivalently taking a vector $x = (0, 0, 1, -1)^T$, which is obviously nonzero vector in $\mathcal{R}^4$, we have $x^T B x = 0$, so matrix B is NOT positive definite.

Matrix C is an SPD matrix.

Solution 1: Let us check all principal minors.

(1) $C_1 = 2 > 0$;

(2) $C_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \quad det(C_2) = 5 > 0$;

(3) $C_3 = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}, \quad det(C_3) = 4 > 0$;

(4) $det(C_4) = det(C) = 3 > 0$.

Solution 2: For a nonzero vector $x = (x_1, x_2, x_3, x_4)^T \in \mathcal{R}^4$ consider

$$(Cx, x) = x^T C x = 2x_1^2 + 2x_2^2 + 2x_3^2 + 2x_4^2 - 2x_1x_2 - 2x_2x_3 - 2x_3x_4.$$

Now use the inequalities

$$-2x_1x_2 \geq -\frac{3}{2}x_1^2 - \frac{2}{3}x_2^2, \quad -2x_2x_3 \geq -x_2^2 - x_3^2, \quad -2x_3x_4 \geq -\frac{3}{2}x_4^2 - \frac{2}{3}x_3^2$$

in the above equality to get

$$(Cx, x) \geq \frac{1}{2}x_1^2 + \frac{1}{3}x_2^2 + \frac{1}{3}x_3^2 + \frac{1}{2}x_4^2 > 0,$$

which shows that the matrix C is positive definite.

Or you can use the representation

$$\begin{aligned}(Cx, x) = x^T Cx &= 2x_1^2 + 2x_2^2 + 2x_3^2 + 2x_4^2 - 2x_1x_2 - 2x_2x_3 - 2x_3x_4 \\ &= x_1^2 + (x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_4)^2 + x_4^2.\end{aligned}$$

Obviously, $(Cx, x) \geq 0$ and $(Cx, x) = 0$ only if $x_1 = x_2 = x_3 = x_4 = 0$, which showes the positive definiteness. This representation has the advantage that it works for C in $\mathcal{R}^{n\times n}$ for any integer n.