MATH 609-600
Homework #2
Vector and matrix norms

This homework is designed so you get familiarized with the concept of matrix norm and the relationships between varuious norms. Solve any set of problems for 100 points.

(1) (20 pts) For a matrix $A \in \mathbf{R}^{n\times n}$ we define a norm by
$$\|A\| = \max_{x\in \mathbf{R}^n, x\neq 0} \|Ax\|/\|x\|,$$
where $\|x\|$ is a norm in $\mathbf{R}^n$. Show that the following is true:
(a) $\|A\| \geq 0$ and if $\|A\| = 0$ then $A = 0$ (here 0 is the zero matrix in $\mathbf{R}^{n\times n}$);
(b) $\|\alpha A\| = |\alpha|\|A\|$, α real number;
(c) $\|A + B\| \leq \|A\| + \|B\|$;
(d) $\|AB\| \leq \|A\|\|B\|$.

(2) (20 pts) Show that a symmetric n by n matrix with positive elements that is strictly row-wise diagonally dominant is also positive definite.

(3) (20 pts) Let $\rho(A)$ be the spectral raduis of the matrix A, that is
$$\rho(A) = \max\{\lambda : \quad \text{where } \lambda \ \text{ is an eigenvalue of } A.\}$$
Show that for any integer $k > 0$, $\rho(A^k) = (\rho(A))^k$.

(4) (10 pts) Prove that $\|A\|_2 \leq \sqrt{n}\|A\|_\infty$ for any matrix $A \in \mathbf{R}^{n\times n}$. Here the matrix norms are subordinate to the corresponding vector norms in $\mathbf{R}^n$: namely,
$$\|x\|_2 = \left(\sum_{i=1}^n x_i^2\right)^{\frac{1}{2}} \text{ (is the Euclidean norm) and } \|x\|_\infty = \max_{i=1}^n |x_i|.$$

(5) (10 pts) Prove that for any nonsingular matrices $A, B \in \mathbf{R}^{n\times n}$
$$\frac{\|B^{-1} - A^{-1}\|}{\|B^{-1}\|} \leq cond(A)\frac{\|B - A\|}{\|A\|},$$
where $\|.\|$ is a matrix norm and $cond(A) = \|A\|\,\|A^{-1}\|$ is the condition number of A with respect to that norm.

(6) (20 pts) Show that any strictly diagonally domonant symmetric matrix with positive diagonal elements has only positive eigenvalues.

(7) (20 pts) Show that if A is symmetric and positive definite matrix and B is symmetric then the eigenvalues of AB are real. If in addition B is positive definite then the eigenvalues of AB are positive.

(8) (20 pts) Show that
$$\text{(a)} \quad \|A\|_\infty = \max_{1\leq i\leq n}\sum_{j=1}^n |a_{ij}|, \quad \text{where} \quad \|x\|_\infty = \max_{1\leq i\leq n}|x_i|;$$
$$\text{(b)} \quad \|A\|_1 = \max_{1\leq j\leq n}\sum_{i=1}^n |a_{ij}|, \quad \text{where} \quad \|x\|_1 = \sum_{i=1}^n |x_i|.$$