2. Solutions

(1) Problem 1:

Proof. (a) If $A = 0$, then cleary $Ax = 0, \forall x \in R^n$, so

$$\|A\| = \max_{x \in R^n, x \neq 0} \frac{\|Ax\|}{\|x\|} = 0.$$

Otherwise $\exists\, a_{ij} \neq 0, a_{ij} \in A$. Let $e_j = [0, \cdots, 1, \cdots, 0]^T$, then

$$\|A\| = \max_{x \in R^n, x \neq 0} \frac{\|Ax\|}{\|x\|} \geq \frac{\|Ae_j\|}{\|e_j\|} > 0.$$

In summary, $\|A\| \geq 0$, and $\|A\| = 0$ iff $A = 0$.

(b)

$$\|\alpha A\| = \max_{x \in R^n, x \neq 0} \frac{\|\alpha Ax\|}{\|x\|} = \max_{x \in R^n, x \neq 0} \frac{|\alpha|\|Ax\|}{\|x\|}$$
$$= |\alpha| \max_{x \in R^n, x \neq 0} \frac{\|Ax\|}{\|x\|} = |\alpha|\|A\|$$

(c)

$$\begin{aligned} \|A + B\| &= \max_{x \in R^n, x \neq 0} \frac{\|(A+B)x\|}{\|x\|} = \max_{x \in R^n, x \neq 0} \frac{\|Ax + Bx\|}{\|x\|} \\ &\leq \max_{x \in R^n, x \neq 0} \frac{\|Ax\| + \|Bx\|}{\|x\|} \\ &\leq \max_{x \in R^n, x \neq 0} \frac{\|Ax\|}{\|x\|} + \max_{x \in R^n, x \neq 0} \frac{\|Bx\|}{\|x\|} = \|A\| + \|B\| \end{aligned}$$

(d) According to the definition of the matrix norm, $\|A\| = \max_{x \in R^n} \|Ax\|/\|x\|$ we have $\|Ax\| \leq \|A\|\|x\|, \forall\, x \in R^n$. So, $\forall\, x \in R^n$ we have the following inequalities:

$$\|ABx\| \leq \|A\|\|Bx\| \leq \|A\|\|B\|\|x\|$$

Therefore, $\|AB\| = \max_{x \in R^n, x \neq 0} \frac{\|ABx\|}{\|x\|} \leq \max_{x \in R^n, x \neq 0} \frac{\|A\|\|B\|x\|}{\|x\|} = \|A\|\|B\|.$

□

(2) Problem 2:

Proof. $A \in R^{n \times n}$ is symmetric, so $\exists$ a real diagonal matrix Λ and an orthonormal matrix Q such that $A = Q\Lambda Q^T$, where the diagonal values of Λ are A's eigenvalues $\lambda_i,\ i = 1, \cdots, n$ and the columns of Q are the corresponding eigenvectors.

According to Gerschgorin's theorem, A's eigenvalues are located in the union of disks

$$d_i = \{z \in C : |z - a_{i,i}| \leq \sum_{j \neq i} |a_{i,j}|\}, \quad i = 1, \cdots, n.$$

Because A is positive and strictly row-wise diagonal dominant, so,

$$d_i = \{z \in C : 0 < a_{i,i} - \sum_{j \neq i} a_{i,j} \leq z \leq \sum_j a_{i,j}\}, \quad i = 1, \cdots, n.$$

Therefore all A's eigenvalues are positive, then $\forall x \in R^n,\ x \neq 0$ we have,

$$x^T Ax = x^T Q\Lambda Q^T x = y^T \Lambda y > 0, \text{ where } y = Q^T x \neq 0.$$

i.e. A is positive definite. □

(3) Problem 3: First, recall that the set of complex numbers $\sigma(A) = \{\lambda :\ det(A - \lambda I) = 0\}$ is called spectrum of A.

Proof. Then, $\forall\ \lambda \in \sigma(A)$, we show that $\lambda^k \in \sigma(A^k)$. Indeed, let $Au = \lambda u$ with $u \neq 0$.

$$A^k u = \lambda A^{k-1} u = \cdots = \lambda^k u.$$

For any $\mu \in \sigma(A^k)$, we have $A^k v = \mu v,\ v \neq 0$. Since, v belong to the range of A, so at least there exists one eigenvalue $\lambda = \mu^{1/k}$ s.t.

$$Av = \mu^{1/k} v.$$

Thus, we always have that for any $|\mu|$ where $A^k u = \mu u,\ u \neq 0$, there exists a $|\lambda| = |\mu|^{1/k}$ where $Au = \lambda u,\ u \neq 0$; for any $|\lambda|$ where $Au = \lambda u,\ u \neq 0$, there exists a $|\mu| = |\lambda^k| = |\lambda|^k$ where $A^k u = \mu u,\ u \neq 0$. So,

$$\rho(A^k) = \max_{\mu \in \sigma(A^k)} |\mu| \equiv \max_{\lambda \in \sigma(A)} |\lambda|^k = (\max_{\lambda \in \sigma(A)} |\lambda|)^k = (\rho(A))^k.$$

□

(4) Problem 4:

Proof. It follows from the obvious string of inequalities

$$\begin{aligned}
\|A\|_2 &= \max_{\|x\|_2=1} \|Ax\|_2 = \max_{\|x\|_2=1} (\sum_{i=1}^n (\sum_{j=1}^n a_{ij} x_j)^2)^{1/2} \\
&\leq \max_{\|x\|_2=1} (\sum_{i=1}^n \sum_{j=1}^n a_{ij}^2 \sum_{j=1}^n x_j^2)^{1/2} \qquad \text{Schwartz inequality} \\
&= (\sum_{i=1}^n \sum_{j=1}^n a_{ij}^2)^{1/2} \leq (\sum_{i=1}^n (\sum_{j=1}^n |a_{ij}|)^2)^{1/2} \\
&\leq (n(\max_i \sum_{j=1}^n |a_{ij}|)^2)^{1/2} \leq (n\|A\|_\infty^2)^{1/2} = \sqrt{n}\|A\|_\infty
\end{aligned}$$

□

(5) Problem 5:

Proof. Assume the norm $\|\cdot\|$ satisfies the submultiplicative property, i.e. $\|AB\| \leq \|A\|\|B\|$ (subordinate matrix norms have this property).

$$\begin{aligned}
\|B^{-1} - A^{-1}\| &= \|A^{-1} - B^{-1}\| = \|B^{-1}(B - A)A^{-1}\| \\
&\leq \|B^{-1}\|\|B - A\|\|A^{-1}\|
\end{aligned}$$

A, B are nonsingular matrices, divide $\|B^{-1}\|$ on both sides,

$$\frac{\|B^{-1} - A^{-1}\|}{\|B^{-1}\|} \leq \|B - A\|\|A^{-1}\| = \frac{\|B - A\|\|A^{-1}\|\|A\|}{\|A\|} = cond(A)\frac{\|B - A\|}{\|A\|}$$

□

(6) Problem 6:

Proof. For any symmetrix matrix A, all its eigenvalues are real.

According to Gerschgorin's theorem, A's eigenvalues are located in the union of disks

$$d_i = \{z \in C : |z - a_{i,i}| \leq \sum_{j \neq i} |a_{i,j}|\}, \quad i = 1, \cdots, n.$$

i.e.,

$$d_i = \{z \in C : a_{i,i} - \sum_{j \neq i} |a_{i,j}| \leq z \leq a_{i,i} + \sum_{j \neq i} |a_{i,j}|\}, \quad i = 1, \cdots, n.$$

Because A's diagonal elements are positive and A is strictly diagonal dominant, so,

$$a_{i,i} - \sum_{j \neq i} |a_{i,j}| = |a_{i,i}| - \sum_{j \neq i} |a_{i,j}| > 0$$

$$a_{i,i} + \sum_{j \neq i} |a_{i,j}| = |a_{i,i}| + \sum_{j \neq i} |a_{i,j}| > 0, \quad i = 1, \cdots, n.$$

So, all this disks are located at the right side of the y-axis in the complex plane, so all eigenvales of A are positive. □

(7) Problem 7:

Proof. First possible solution:

(1) Consider an eigenvalue λ and its eigenvector ψ of the matrix AB, that is $AB\psi = \lambda\psi$. We do not know whether λ and ψ are real, so we assume that they are complex. Recal that the inner product of two complex vectors ψ and ϕ is defined as $(\phi, \psi) = \sum_i \phi_i \bar{\psi}_i$, where $\bar{\psi}$ is the complex conjugate to ψ. Recall, that $(\phi, \psi) = \overline{(\psi, \phi)}$. For the inner product of the complex vectors $AB\psi$ and ψ we get

$$AB\psi = \lambda\psi \quad \Rightarrow BAB\psi = \lambda B\psi, \quad \Rightarrow (BAB\psi, \psi) = \lambda(B\psi, \psi).$$

Now using the symmetry of A and B we get

$$(BAB\psi, \psi) = (\psi, BAB\psi) = \overline{(BAB\psi, \psi)},$$

which means that the complex number $(BAB\psi, \psi)$ is equal to its complex conjugate, i.e. the number is real. In the same way we prove that $(B\psi, \psi)$ is reas as well, from where we conclude that λ is real. Then we conclude that ψ is real as well.

(2) If A and B are positive definite, then $(B\psi, \psi) > 0$ and $(BAB\psi, \psi) > 0$, therefore from $(BAB\psi, \psi) = \lambda(B\psi, \psi)$ it follows that $\lambda > 0$.

Another possible solution (for those with more advanced knowlegde in linear algebra):

(1) A is SPD, its square root exists $A^{1/2}$ which is SPD, then

$$AB \sim A^{-1/2}ABA^{1/2} = A^{1/2}BA^{1/2}$$

Because both $A^{1/2}$ and B are symmetric, then

$$(A^{1/2}BA^{1/2})^T = (A^{1/2})^T B^T (A^{1/2})^T = A^{1/2}BA^{1/2},$$

so $A^{1/2}BA^{1/2}$ is symmetric, and the eigenvalues of $A^{1/2}BA^{1/2}$ are real. $AB \sim A^{1/2}BA^{1/2}$, they have the same spectrum, so all eigenvalues of AB are real.

(2) If B is also SPD, we can show that $A^{1/2}BA^{1/2}$ is SPD. The symmetry is shown in (1). Now show positive definite. $\forall x \in R^n, x \neq 0$ with $y = A^{1/2}x \neq 0$ we have

$$(A^{1/2}BA^{1/2}x, x) = (BA^{1/2}x, A^{1/2}x) = (By, y) > 0.$$

So $A^{1/2}BA^{1/2}$ is SPD, and all its eigenvalues are positive. $AB \sim A^{1/2}BA^{1/2}$, they have the same spectrum, so all eigenvalues of AB are positive.

□

we show (b) as well

(8) <u>Problem 8:</u>

Proof. We prove (a). First we show that $\|A\|_\infty \leq \max_i \sum_j |a_{i,j}|$. Indeed,

$$\begin{aligned}
\|A\|_\infty &= \max_{\|x\|_\infty = 1} \|Ax\|_\infty \\
&= \max_{\|x\|_\infty = 1} \max_i |\sum_{j=1}^n a_{ij}x_j| \\
&\leq \max_{\|x\|_\infty = 1} \max_i \sum_{j=1}^n |a_{ij}||x_j| \\
&\leq \max_{\|x\|_\infty = 1} \max_i \max_j |x_j| \sum_{j=1}^n |a_{ij}| \\
&= \max_i \sum_{j=1}^n |a_{ij}|
\end{aligned}$$

Next we show aslo that $\max_i \sum_j |a_{i,j}| \leq \|A\|_\infty$. Assume that for some integer $k \in [1, n]$ we have

$$\sum_{j=1}^n |a_{kj}| = \max_i \sum_{j=1}^n |a_{ij}|.$$

By choosing a vector y s.t.,

$$y_j = \begin{cases} 1, & \text{for } a_{k,j} \geq 0; \\ -1, & \text{for } a_{k,j} < 0; \end{cases}$$

Then, $\|y\|_\infty = 1$ and

$$\|A\|_\infty = \sup_{x \in \mathcal{R}^n, \|x\|_\infty = 1} \|Ax\|_\infty \geq \|Ay\|_\infty = \sum_j |a_{k,j}| = \max_i \sum_j |a_{i,j}|.$$

Thus, from the inequalities $\max_i \sum_j |a_{i,j}| \leq \|A\|_\infty \leq \max_i \sum_j |a_{i,j}|$, it follows that

$$\|A\|_\infty = \max_i \sum_j |a_{i,j}|.$$

The inequality (b) is shown in a similar way. □