HOMEWORK 3

DUE: MONDAY, 21 OCTOBER 2013

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Solve the problems below, worth 20 points each, for a total of 100 points. Fully justify your work.

(1) Derive an expression for the zeros of the minimizing polynomial

$$P_m(x) = \frac{T_m(L(x))}{T_m(L(0))}$$

Here T_m is the m^{th} Chebyshev polynomial and

$$L(\lambda) = -1 + \frac{2}{\lambda_M - \lambda_0}(\lambda - \lambda_0)$$

(Note that there is no matrix involved in this problem, but on our applications, this polynomial was important for a matrix with smallest eigenvalue λ_0 and large eigenvalue λ_M.)

(2) Let $A \in \mathbb{R}^{n\times n}$ be a symmetric positive definite. Show that, for any polynomial $P(\cdot)$,

$$\|P(A)\|_A = \rho(P(A))$$

for any polynomial P. (Hint: use the definition of the matrix norm and expand everything out in terms of the $(\cdot,\cdot)$-orthonormal eigenvectors of A).

(3) In many examples, we have seen the $n \times n$ matrix

$$D = \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}$$

Show that D has eigenvalues $\lambda_j = 4\sin^2\left(\frac{\pi j}{2(n+1)}\right)$ corresponding to eigenvectors ψ_j, whose k^{th} component is given by

$$(\psi_j)_k = \sin\left(\frac{\pi j k}{n+1}\right).$$

(Hint: Use a basic trig identities.) In the case $n = 100$, plot the smallest eigenvector and the largest eigenvector (i.e., make the x-axis the component number, and the y-axis the value of the k^{th} component).

(4) Let A be an SPD matrix and consider the problem $A\mathbf{x} = \mathbf{b}$. Suppose instead of minimizing the quadratic functional $q(\mathbf{x}) = (A\mathbf{x}, \mathbf{x}) - 2(\mathbf{b}, \mathbf{x})$ to solve this problem, we instead tried to minimize

$$J(\mathbf{x}) = \frac{1}{2}\|A\mathbf{x} - \mathbf{b}\|_{L^2}^2$$

where $\|\cdot\|_{L^2}$ is the standard Euclidean norm. What can you say about such an approach? Will this yield a useful method for solving $A\mathbf{x} = \mathbf{b}$?

(5) Given a linearly-independent set of vectors $\{\mathbf{r}_0, \mathbf{r}_1, \dots, \mathbf{r}_k\}$ and an SPD matrix A, show that one can construct an A-orthonormal set of vectors $\{\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_k\}$ such that

$$\text{span}(\{\mathbf{r}_0, \mathbf{r}_1, \dots, \mathbf{r}_k\}) = \text{span}(\{\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_k\}).$$

Recall that being A-orthonormal means that $(A\mathbf{p}_i, \mathbf{p}_j) = \delta_{ij}$. (Hint: Start by choosing $\mathbf{p}_0$ as a multiple of $\mathbf{r}_0$. Then assume all $\mathbf{p}_m$ have been chosen, and show how to construct $\mathbf{p}_{m+1}$.)