MATH609 – 600

Homework #3

Problems and possible solutions

Due October 4, 2011

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1. Problems

This homework is designed so you get familiarized with the concept of the simplest iterative methods and the tools for their study. Solve any set of problems for 100 points.

Below we use the following notations: $A \in \mathbb{R}^{n\times n}$ means a n $n \times n$ square matrix with real elements; $x \in \mathbb{C}^n$ means a vector x with entries complex numbers; for $x, y \in \mathbb{C}$ the Euclidean inner product is defined by $(x, y) = \sum_{j=1}^{n} x_j \bar{y}_j$, where $\bar{y}_j$ is the complex conjugate to y_j.

(1) (10 pts) Prove that if A is nonsingular matrix with real elements, then $A^t A$ is positive definite.

(2) (10 pts) Prove that if A is a real matrix and such that (Ax, x) is real and positive for any nonzero vector $x \in \mathbb{C}^n$, then its eigenvalues are positive.

(3) (10 pts) Let $\rho(A)$ be the spectral radius of the matrix A. Show that for any integer number $k > 0$, $\rho(A^k) = (\rho(A))^k$.

(4) (10 pts) Is there a matrix A such that $\rho(A) < \|A\|$ for all subordinate matrix norm ?

(5) (20 pts) Prove that if A is unit column diagonally dominant, that is

$$a_{jj} = 1 > \sum_{i\neq j} |a_{ij}|, \quad 1 \le j \le n,$$

then Jacobi iteration is convergent. Similarly, if A is unit row diagonally dominant, then the Jacobi iterartion converges as well.

(6) (20 pts) Prove that if A is nonsingular and if $|\lambda| < \|A^{-1}\|^{-1}$, then λ is not an eigenvalue of A. Here the norm can be any subordinate matrix norm.

(7) (40 pts) **This is a challenging one; but do not worry, we'll consider it in more generality in class** Consider $n\times n$ variant of the matrix C of Problem # 7 of your HW#1. This matrix has eigenvalues $\lambda_j = 4\sin^2 \frac{\pi j}{2(n+1)}$, $j = 1, 2, \dots, n$. Find the iteration parameters τ_1 and τ_2 in the following two-stage method for solving the system $Cx = b$:

$$x^{(2k+1)} = (I - \tau_1 C)x^{(2k)} + \tau_1 b, \quad x^{(2k+2)} = (I - \tau_2 C)x^{(2k+1)} + \tau_2 b,$$

for $k = 0, 1, 2, \dots$ so that the method converges the fastest possible way.

(8) (20 pts) Prove that if $\rho(A) < 1$, then the matrix $I - A$ is invertible and

$$(I - A)^{-1} = \sum_{k=0}^{\infty} A^k.$$

(9) (10 pts) Prove that $\rho(A) < 1$ if and only if $\lim_{k\to\infty} A^k x = 0$ for every $x \in \mathbb{C}^n$.

(10) (20 pts) Assume that the matrix A is split in the following way $A = D - L - U$, where D is diagonal matrix and L and U are strictly lower and upper triangular matrices, respectively. Consider the iteration method in the form $x^{k+1} = Gx^k + c$, where G is the iteration matrix. Find the iteration matrices G of the backward SOR and the SSOR method. SSOR is defined as one forward SOR sweep followed by one backward SOR sweep.

2. Solutions

Problem 1

Prove that if A is nonsingular matrix with real elements, then $A^T A$ is positive definite.

Solution: A matrix $A \in \mathbb{R}^{n\times n}$ is called positive definite if

$$x^T Ax > 0, \; \forall x \in \mathbb{R}^n, x \neq 0.$$

Here, $A \in \mathbb{R}^{n\times n}$ and nonsingular. $A^T A$ is real, and $(A^T A)^t = A^T A$, is symmetric. $\forall x \in R^n, x \neq 0$ we have,

$$(A^T Ax, x) = (Ax, Ax) = (y, y) > 0, \text{ where } y = Ax \neq 0.$$

So $A^T A$ is symmetric positive definite. This completes the proof.

Problem 2

Prove that if A is a real matrix and such that (Ax, x) is real and positive for any nonzero vector $x \in \mathbb{C}^n$, then its eigenvalues are positive.

Solution: For any eigenpair $\{\lambda \in \mathbb{C}, u \in \mathbb{C}^n\}$ of A we have,

$$Au = \lambda u, \; u \neq 0.$$

Because (Au, u) is real and positive for $u \in \mathbb{C}^n, u \neq 0$, then $(Au, u) = \lambda(u, u)$ with $(Au, u) > 0$ and $(u, u) = \|u\|^2 > 0$ implies that λ is real and positive.

Since λ is arbitrary, then all eigenvalues of A are positive.

Problem 3

Let $\rho(A)$ be the spectral radius of the matrix A. Show that for any integer number $k > 0$, $\rho(A^k) = (\rho(A))^k$.

Solution: First, recall that the set of complex numbers $\sigma(A) = \{\lambda : \; det(A - \lambda I) = 0\}$ is called spectrum of A.

Then, $\forall \; \lambda \in \sigma(A)$, we show that $\lambda^k \in \sigma(A^k)$. Indeed, let $Au = \lambda u$ with $u \neq 0$.

$$A^k u = \lambda A^{k-1} u = \cdots = \lambda^k u.$$

For any $\mu \in \sigma(A^k)$, we have $A^k v = \mu v, \; v \neq 0$. Since, v belong to the range of A, so at least there exists one eigenvalue $\lambda = \mu^{1/k}$ s.t.

$$Av = \mu^{1/k} v.$$

Thus, we always have that for any $|\mu|$ where $A^k u = \mu u, \; u \neq 0$, there exists a $|\lambda| = |\mu|^{1/k}$ where $Au = \lambda u, \; u \neq 0$; for any $|\lambda|$ where $Au = \lambda u, \; u \neq 0$, there exists a $|\mu| = |\lambda^k| = |\lambda|^k$ where $A^k u = \mu u, \; u \neq 0$. So,

$$\rho(A^k) = \max_{\mu \in \sigma(A^k)} |\mu| \equiv \max_{\lambda \in \sigma(A)} |\lambda|^k = (\max_{\lambda \in \sigma(A)} |\lambda|)^k = (\rho(A))^k.$$

Problem 4

Is there a matrix A such that $\rho(A) < \|A\|$ for all subordinate matrix norm ?

Solution: Yes. For G-HW-3-sol.tex example,

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$\rho(A) = 0$, it is less than any subordinate matrix norm of A (any norm of A is larger that zero).

In fact, all nonzero upper or lower triangular matrices with zero diagonal elements have $\rho(A) = 0$, while any subordinate norm will be greater than 0.

Problem 5

Prove that if A is unit column diagonally dominant, that is

$$a_{jj} = 1 > \sum_{i \neq j} |a_{ij}|, \quad 1 \leq j \leq n,$$

then Jacobi iteration is convergent.

Solution: Jacobi iteration in this case, since $D = I$, is,

$$x^k = (I - A)x^{k-1} + b$$

By the Fundamental Theorem for iterative methods we know that the iteration converges if and only if $\rho(I - A) < 1$.

Now you can use a variant of Gerschgorin theorem saying that all eigenvalues of $I - A$ are inside the union of the disks (can you prove this one ???),

$$D_j = \{z \in \mathbb{C}^n : |z - (1 - a_{jj})| \leq \sum_{i \neq j} |a_{i,j}|\}, \ j = 1, \cdots, n.$$

A is unit column diagonally dominant, that is

$$a_{jj} = 1 > \sum_{i \neq j} |a_{ij}|, \ j = 1, \cdots, n.$$

so,

$$D_j = \{z \in \mathbb{C}^n : |z| \leq \sum_{i \neq j} |a_{ij}| < 1\}, \ j = 1, \cdots, n.$$

Therefore, all eigenvalues are inside the unit circle in the complex plane, so $\rho(I - A) < 1$, and Jacobi iteration is convergent.

Another Solution: We know (from HW#2, for example) that for $G \in \mathbb{R}^{n \times n}$ we have $\|G\|_1 = \max_j \sum_i |g_{ij}|$. Then obviously

$$\|I - A\|_1 = \max_j \sum_{i \neq j} |a_{ij}| = \delta < 1.$$

Therefore, the Jacobi iteration converge and the following estimate is valid for the error $e^{(k)} = x^{(k)} - x$:

$$\|e^{(k)}\| \leq \delta^k \|e^{(0)}\|.$$

Problem 6

Prove that if A is nonsingular and if $|\lambda| < \|A^{-1}\|^{-1}$, then λ is not an eigenvalue of A. Here the norm can be any subordinate matrix norm.

Solution: For any eigenvalue μ of A we have

$$Au = \mu u,\ u \neq 0$$

A is nonsingular, so $\mu \neq 0$. Multiplying $\mu^{-1}A^{-1}$ on both sides of the equation above, we get

$$\mu^{-1}u = A^{-1}u,\ u \neq 0$$

This shows that μ^{-1} is an eigenvalue of A^{-1}.

For any subordinate matrix norm $\|\cdot\|$ we have,

$$\|A^{-1}\| = \max_{x\neq 0} \frac{\|A^{-1}x\|}{\|x\|} \geq \frac{\|A^{-1}u\|}{\|u\|} = \frac{\|\mu^{-1}u\|}{\|u\|} = |\mu^{-1}|$$

i.e., $|\mu| \geq \|A^{-1}\|^{-1}$. μ is an arbitrary eigenvalue, therefore, if $|\lambda| < \|A^{-1}\|^{-1}$, it cannot be an eigenvalue of A.

Problem 7

Consider $n \times n$ variant of the matrix C of Problem # 7 of your HW#1. This matrix has eigenvalues $\lambda_j = 4\sin^2 \frac{\pi j}{2(n+1)}$, $j = 1, 2, \ldots, n$. Find the iteration parameters τ_1 and τ_2 in the following two-stage method for solving the system $Cx = b$:

$$x^{(2k+1)} = (I - \tau_1 C)x^{(2k)} + \tau_1 b, \quad x^{(2k+2)} = (I - \tau_2 C)x^{(2k+1)} + \tau_2 b,$$

for $k = 0, 1, 2, \ldots$ so that the method converges the fastest possible way.

One possible solution: One can eliminate the iteration of odd number to get

$$x^{(2k+2)} = (I - \tau_2 C)((I - \tau_1 C)x^{(2k)} + \tau_1 b) + \tau_2 b,$$

which could be written for the error $e^{(2k)} = x^{(2k)} - x$ in the form

$$e^{(2k+2)} = (I - \tau_2 C)(I - \tau_1 C)e^{(2k)} := Ge^{(2k)}$$

Since C is a symmetric matrix the G is symmetric and we know that $\rho(G) = \|G\|_2$. For convergence it is sufficient (and in this case necessary) to have $\|G\|_2 = \delta < 1$. To get the fastest possible convergence we need to find such τ_1 and τ_2 that

$$\|(I - \tau_2 C)(I - \tau_1 C)\|_2 = \max_{j=1,\ldots,n} |(1 - \tau_2\lambda_j)(1 - \tau_1\lambda_j)| \quad \text{is the smallest possible.}$$

This is the same as to find such τ_1, τ_2 so that function $f(\lambda) = (1 - \tau_2\lambda)(1 - \tau_1\lambda)$ has the smallest maximum of its absolute value in the interval $\lambda_1 \leq \lambda \leq \lambda_n$.

Obviously, for fixed τ_1 and τ_2 the function $f(\lambda)$ is a parabola and the extremal values of $|f(\lambda)|$ are at the following three points $\lambda = \lambda_1, \frac{\lambda_1+\lambda_2}{2}, \lambda_2$. Minimizing with respect to τ_1 and τ_2 will lead to the following two nonlinear equations for these two parameters:

$$f(\lambda_1) = f(\lambda_2) = -f(\frac{\lambda_1 + \lambda_2}{2}).$$

Taking into account that $\lambda_1 = 4\sin^2 \frac{\pi}{2(n+1)}$, $\lambda_2 = 4\cos^2 \frac{\pi}{2(n+1)}$, and $\frac{\lambda_1+\lambda_2}{2} = 2$ we can solve the system and get the following approximate solution:

$$\frac{1}{\tau_1} = 2 - \sqrt{2}\sqrt{1-\lambda_1} \quad \frac{1}{\tau_2} = 2 + \sqrt{2}\sqrt{1-\lambda_1}$$

Solving exactly the system is possible, but there is much more elegant general construction we shall consider in class. Just for the sake of exercise here I have neglected the terms that involve λ_1^2. This is possible only for large n since $\lambda_1^2 = O(n^{-4})$.

Obviously, a good approximation of the iterartion parameters are the following values:

$$\tau_1 = 1/(2-\sqrt{2}) \quad \tau_2 = 1/(2+\sqrt{2})$$

Problem 8

Prove that if $\rho(A) < 1$, then the matrix $I - A$ is invertible and

$$(I-A)^{-1} = \sum_{k=0}^{\infty} A^k.$$

Solution: Since $\rho(A) < 1$, then $det(I-A) \neq 0$, so $I - A$ is invertible (nonsingular).

$\forall \delta > 0, \exists \|\cdot\|_\delta$, a matrix norm subordinate to a vector norm, s.t. $\|A\|_\delta \leq \rho(A) + \delta$. By assumption, $\rho(A) < 1$, take $\delta = 1/2(1-\rho(A))$, then $\|A\|_\delta \leq 1/2(1+\rho(A)) < 1$.

$$(I-A)\sum_{k=0}^{m} A^k = \sum_{k=0}^{m}(A^k - A^{k+1}) = I - A^{m+1}$$

Since $\|A^{m+1}\|_\delta \leq \|A\|_\delta^{m+1} \to 0$ as $m \to \infty$. Thus,

$$(I-A)\sum_{k=0}^{m} A^k \longrightarrow I, \text{ as } m \to \infty$$

i.e.

$$(I-A)\sum_{k=0}^{\infty} A^k = I.$$

Multiply $(I-A)^{-1}$ on both sides of the equation above, we get,

$$(I-A)^{-1} = \sum_{k=0}^{\infty} A^k.$$

Problem 9

Prove that $\rho(A) < 1$ if and only if $\lim_{k\to\infty} A^k x = 0$ for every $x \in \mathbb{C}^n$.

Solution:

$\Longleftarrow$ $\rho(A) < 1$, according to the result of Problem 8, we know that $I - A$ is nonsingular and

$$(I-A)^{-1} = \sum_{k=0}^{\infty} A^k.$$

Multiply any vector $x \in \mathbb{C}^n$ on both sides of the equation above,

$$(I-A)^{-1}x = \sum_{k=0}^{\infty} A^k x.$$

So $\{A^k x\},\ k = 0, 1, \cdots, \infty$ is a convergent series, i.e., $\lim_{k\to\infty} A^k x = 0$.

$\Longrightarrow \lim_{k\to\infty} A^k x = 0, \forall x \in \mathbb{C}^n$:
$\forall \lambda \in \mathbb{C},\ u \in \mathbb{C}^n$ s.t. $Au = \lambda u$, we have $0 = \lim_{k\to\infty} A^k u = \lim_{k\to\infty} \lambda^k u$, so $\lambda < 1$. Since λ is arbitrary, therefore,

$$\rho(A) = \max_{det(I-\lambda A)=0} |\lambda| < 1.$$

Problem 10

Assume that the matrix A is split in the following way $A = D - L - U$, where D is diagonal matrix and L and U are strictly lower and upper triangular matrices, respectively. Consider the iteration method in the form $x^{k+1} = Gx^k + c$, where G is the iteration matrix. Find the iteration matrices G of the backward SOR and the SSOR method. SSOR is defined as one forward SOR sweep followed by one backward SOR sweep.

Solution: During $k+1$ step, the backward SOR calculate the new $x^{(k+1)}$ this way,

$$\xi_i = (-\sum_{j<i} a_{ij} x_j^{(k)} - \sum_{j>i} a_{ij} x_j^{(k+1)} + b)/a_{ii}$$
$$x_i^{(k+1)} = \omega \xi_i + (1-\omega) x_i^{(k)}$$

In matrix form,

$$Dx^{(k+1)} = \omega(Lx^{(k)} + Ux^{(k+1)} + b) + (1-\omega)Dx^{(k)}$$

Written in the $x^{(k+1)} = Gx^{(k)} + c$ format,

$$x^{(k+1)} = (D - \omega U)^{-1}(\omega L + (1-\omega)D)x^{(k)} + (D - \omega U)^{-1}\omega b$$

So the backward SOR iteration matrix $G_{bSOR} = (D - \omega U)^{-1}(\omega L + (1-\omega)D)$.

In the same way we can find the forward SOR iteration matrix $G_{fSOR} = (D - \omega L)^{-1}(\omega U + (1-\omega)D)$. For SSOR, each step is a combination of a forward SOR followed by a backward SOR, so,

$$\begin{aligned} G_{SSOR} &= G_{bSOR} G_{fSOR} \\ &= (D - \omega U)^{-1}(\omega L + (1-\omega)D)(D - \omega L)^{-1}(\omega U + (1-\omega)D) \end{aligned}$$
