MATH 609-600
Homework # 4
Polynomial interpolation of functions of one variable

Solve any set of problems for 100 points. The homework should be presented at the beginning of the class. There penalty for delay of the homework is 5 pts per day.

(1) (20 pts) Find the Lagrange and backward Newton divided difference interpolating polynomials for the data $(0,1),\ (0.5,2),\ (1,3),\ (1.5,4)$.

(2) (20 pts) Estimate the interpolation error of $\cos x$ in the interval $(0, 0.4)$ by a polynomial of degree 2 using the interpolation nodes $x_0 = 0,\ x_1 = 0.2,\ x_2 = 0.4$.

(3) (20 pts) Prove the identities $\sum_{k=0}^{n}(x - x_k)^m l_{n,k}(x) = 0$ for $m = 1, ..., n$. The basic polynomials $l_{n,k}(x)$ were defined in class as

$$l_{n,k}(x) = \frac{(x - x_0)\dots(x - x_{k-1})(x - x_{k+1})\dots(x - x_n)}{(x_k - x_0)\dots(x_k - x_{k-1})(x_k - x_{k+1})\dots(x_k - x_n)}.$$

(4) Let $f(x) = x^n$ and $f[x_0, x_1, ..., x_n]$ be the divided difference of order n using the points $x_0 < x_1 < ... < x_n$. Prove that:
 (a) (10 pts) $f[x_0, x_1, ..., x_n] = 1$;
 (b) (10 pts) $f[x_0, x_1, ..., x_{n-1}] = x_0 + x_1 + ... + x_{n-1}$.

(5) (10 pts) If $f[x_0, x_1, \dots, x_n]$ denotes the divided difference of order n prove the **Leibnitz formula**

$$(fg)[x_0, x_1, \dots, x_n] = \sum_{k=0}^{n} f[x_0, x_1, \dots, x_k]g[x_k, x_{k+1}, \dots, x_n].$$

(6) (20 pts) Let $p(x)$ be the Hermite intepolating polynomial based on $n+1$ distinct points $x_0 < \cdots < x_n$ in $[a, b]$. Assume that the data $f_i = f(x_i)$ and $f_i' = f'(x_i)$ is generated by a function $f(x) \in C^{(2n+2)}([a, b])$. Prove that for each $x \in [a, b]$ there is a point ξ_x such that

$$f(x) = p(x) + \frac{1}{(2n+2)!}f^{(2n+2)}(\xi_x)(x - x_0)^2 \dots (x - x_n)^2.$$

(7) (20 pts) Prove that there is unique polynomial of degree at most $n + 2$ that interpolates the data: $f(x_0), f'(x_0), f(x_1), f(x_2), \dots, f(x_{n-1}), f(x_n), f'(x_n)$ at the n distinct points $x_0 < \cdots < x_n$.

(8) In your free time and for your amusement(!!!): Show that if

$$\omega(x) = (x - x_0)(x - x_1)...(x - x_n)$$

then:
 (a) $\sum_{k=0}^{n}(x - x_k)^{n+1} l_{n,k}(x) = (-1)^n \omega(x)$
 (b) $\sum_{k=0}^{n}(x - x_k)^{n+2} l_{n,k}(x) = (-1)^n \omega(x) \sum_{k=0}^{n}(x - x_k)$
 (c) $\sum_{k=0}^{n} l_{n,k}(0) x_k^{n+1} = (-1)^n x_0.x_1...x_n$.