MATH 609-600
Homework #5
Numerical Integration & ODE Methods

Solve any set of problems for 100 points. 5 pts per day penalty for delay of the homework applies.

(1) (10 pts) Show that the quadrature

$$\int_0^\infty e^{-x} f(x)dx \approx \frac{2+\sqrt{2}}{4} f(2-\sqrt{2}) + \frac{2-\sqrt{2}}{4} f(2+\sqrt{2})$$

has algebraic degree of accuracy 3.

(2) (10 pts) Find the nodes and the coefficients of the Gauss quadrature with two nodes for evaluating the integral

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}} dx.$$

(3) (20 pts) Prove that if the interval is symmetric with respect to the origin and if $w(x)$ is an even function, then the Gaussian nodes will be symmetric respect to the origin. So if the roots are ordered $x_0 < x_1 < \cdots < x_n$, then $x_i = -x_{n-i}$ and $A_i = A_{n-i}$ for $i = 0, 1, \ldots, n$.

(4) (20 pts) Define the Legendre polynomial $P_n(x)$ of degree n by

$$P_n(x) = \frac{1}{2^n\; n!} \frac{d^n (x^2-1)^n}{dx^n}, \; n = 0, 1, \ldots.$$

Show that $P_n(x)$ has n distinct zeros in the interval $(-1, 1)$ which are symmetric with respect to the origin.

(5) (20 pts) Consider the Gaussian quadrature

$$\int_{-1}^{1} f(x)dx \approx \sum_{i=1}^{n} A_k f(x_k).$$

Show that $A_k = 2/[(1-x_k^2)[P_n'(x_k)]^2], \; k = 1, 2, \ldots, n,$where P_n is the defined above Legendre polynomial.

Hint: Use the fact that $P_n(1) = 1$, $P_n(-1) = (-1)^n$ and the equality

$$\int_{-1}^{1} P_n(x)P_n'(x)/(x - x_k)dx = A_k[P_n'(x_k)]^2.$$

Below we consider the following initial value problem: find $x(t)$ such that $x'(t) = f(t, x)$ for $t > t_0$ and satisfying the initial condition $x(t_0) = x_0$. Also η_n denotes the approximation of $x(t_n)$ by the numerical method.

(6) (20 pts) Consider the Runge-Kutta method

$$\eta_0 = x_0, \; \eta_{i+1} = \eta_i + h\,\Phi(t_i, \eta_i; h), \; i = 0, 1, ... \tag{1}$$

where $\Phi(t, x; h) = \frac{1}{4} k_1(t, x) + \frac{3}{4} k_2(t, x)$ with $k_1(t, x) = f(t, x)$, $k_2(t, x) = f(t + \frac{2}{3}h, x + \frac{2}{3}hk_1)$. Show that the method is of second order.

(7) (20 pts) For the above initial value problem consider the following (in general, implicit) Runge-Kutta method (1), where:

$$\begin{aligned} \Phi(t, x; h) &= a_1 k_1 + a_2 k_2 \\ k_1 &= f(t + \alpha_1 h, x + \beta_{11} h k_1) \\ k_2 &= f(t + \alpha_2 h, x + \beta_{21} h k_1 + \beta_{22} h k_2). \end{aligned} \tag{2}$$

(a) Find the conditions that the coefficients a, α, β need to satisfy so that the **explicit** method (i.e. $\beta_{11} = \beta_{22} = 0$) is of second order. Give at least one set of coefficients that satisfy these conditions.

(b) Find the conditions that the coefficients a, α, β need to satisfy so that the **implicit** method (i.e. $\beta_{11} \neq 0$ and $\beta_{22} \neq 0$) is of second order.

(8) (20 pts) Derive an explicit multi-step method of order four (Adams-Bashforth four step method) that uses integration in the interval (t_i, t_{i+1}). Write down the expression for the local truncation error.