MATH609 – 600

Homework #5

Problems and possible solutions

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1. Problems

This homework is designed so you get familiarized with the concept of the simplest polynomial interpolation formulas and the tools for their study. Below we use the following standard notations:

$$l_{n,k}(x) := L_{n,k} := (x)\frac{(x-x_0)\dots(x-x_{k-1})(x-x_{k+1})\dots(x-x_n)}{(x_k-x_0)\dots(x_k-x_{k-1})(x_k-x_{k+1})\dots(x_k-x_n)}$$

and $q_k(x)=(x-x_0)\dots(x-x_{k-1})$ for $k=0,1,\dots,n+1$ and $\omega(x)=(x-x_0)\dots(x-x_n)$.

Solve any set of problems for 100 points.

(1) (20 pts) Find the Lagrange and backward Newton divided difference interpolating polynomials for the data $(0,1),\ (0.5,2),\ (1,3),\ (1.5,4)$.

(2) (20 pts) Estimate the interpolation error of $\cos x$ in the interval $(0,0.4)$ by a polynomial of degree 2 using the interpolation nodes $x_0=0,\ x_1=0.2,\ x_2=0.4$.

(3) (20 pts) Prove the identities $\sum_{k=0}^n (x-x_k)^m l_{n,k}(x)=0$ for $m=1,...,n$.

(4) Let $f(x)=x^n$ and $f[x_0,x_1,...,x_n]$ be the divided difference of order n using the points $x_0<x_1<...<x_n$. Prove that:
   (a) (10 pts) $f[x_0,x_1,...,x_n]=1$;
   (b) (10 pts) $f[x_0,x_1,...,x_{n-1}]=x_0+x_1+...+x_{n-1}$.

(5) (10 pts) If $f[x_0,x_1,\dots,x_n]$ denotes the divided difference of order n prove the **Leibnitz formula**

$$(fg)[x_0,x_1,\dots,x_n]=\sum_{k=0}^n f[x_0,x_1,\dots,x_k]g[x_k,x_{k+1},\dots,x_n].$$

(6) (20 pts) Let $p(x)$ be the Hermite intepolating polynomial based on $n+1$ distinct points $x_0<\cdots<x_n$ in $[a,b]$. Assume that the data $f_i=f(x_i)$ and $f_i'=f'(x_i)$ is generated by a function $f(x)\in C^{(2n+2)}([a,b])$. Prove that for each $x\in[a,b]$ there is a point ξ_x such that

$$f(x)=p(x)+\frac{1}{(2n+2)!}f^{(2n+2)}(\xi_x)(x-x_0)^2\dots(x-x_n)^2.$$

(7) (20 pts) Prove that there is unique polynomial of degree at most $n+2$ that interpolates the data: $f(x_0),f'(x_0),f(x_1),f(x_2),\dots,f(x_{n-1}),f(x_n),f'(x_n)$ at the n distinct points $x_0<\cdots<x_n$.

(8) In your free time and for your amusement(!!!): Show that for

$$\omega(x)=(x-x_0)(x-x_1)...(x-x_n)$$

we have the following identities:
   (a) $\displaystyle\sum_{k=0}^n (x-x_k)^{n+1} l_{n,k}(x)=(-1)^n\omega(x)$
   (b) $\displaystyle\sum_{k=0}^n (x-x_k)^{n+2} l_{n,k}(x)=(-1)^n\omega(x)\sum_{k=0}^n (x-x_k)$
   (c) $\displaystyle\sum_{k=0}^n l_{n,k}(0)x_k^{n+1}=(-1)^n x_0.x_1...x_n.$

2. Solutions

Problem 1

Find the Lagrange and backward Newton divided difference interpolating polynomials for the data $(0,1),\ (0.5,2),\ (1,3),\ (1.5,4)$.

<u>Solution:</u> The Lagrange interpolation polynomial for point x_i is $l_{n,j}(x)$. Then for the given data $(0,1),(0.5,2),(1,3),(1.5,4)$ we have:

$$
\begin{aligned}
P_{Lagrange}(x) &= 1\,l_{3,0}(x) + 2\,l_{3,1}(x) + 3\,l_{3,2}(x) + 4\,l_{3,3}(x) \\
&= \frac{(x-0.5)(x-1)(x-1.5)}{(0-0.5)(0-1)(0-1.5)} + 2\frac{(x-0)(x-1)(x-1.5)}{(0.5-0)(0.5-1)(0.5-1.5)} \\
&\quad +3\frac{(x-0)(x-0.5)(x-1.5)}{(1-0)(1-0.5)(1-1.5)} + 4\frac{(x-0)(x-0.5)(x-1)}{(1.5-0)(1.5-0.5)(1.5-1)} \\
&= -\tfrac{4}{3}(x-0.5)(x-1)(x-1.5) + 8x(x-1)(x-1.5) \\
&\qquad -12x(x-0.5)(x-1.5) + \tfrac{16}{3}x(x-0.5)(x-1) = 1 + 2x.
\end{aligned}
$$

The backward Newton divided difference interpolating polynomial for the data $(0,1),(0.5,2),(1,3),(1.5,4)$ is:

0	**1**	**2**	**0**	**0**
0.5	2	2	0	
1	3	2		
1.5	4			

so the interpolating polynomial is

$$P_{Newton}(x) = 1 + 2x + 0\,x(x-0.5) + 0\,x(x-0.5)(x-1) = 1 + 2x$$

Problem 2

Estimate the interpolation error of $\cos x$ in the interval $(0, 0.4)$ by a polynomial of degree 2 using the interpolation nodes $x_0 = 0,\ x_1 = 0.2,\ x_2 = 0.4$.

<u>Solution</u>: The theorem on **polynomial interpolation error** states:

> Let $f(x) \in C^{n+1}[a,b]$ and $p(x)$ be the polynomial of degree at most n that interpolates $f(x)$ at $n+1$ distinct points $a \le x_0 < x_1 < \cdots < x_n \le b$. Then for any $x \in [a,b], \exists \xi_x \in (a,b)$, such that
>
> $$f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi_x)\omega(x).$$

Here we use a polynomial degree of 2 to interpolate $\cos(x)$ in the interval $(0, 0.4)$, then we have

$$\begin{aligned}
\cos(x) - p_2(x) &= \tfrac{1}{3!}\cos^{(3)}(\xi_x)(x-x_0)(x-x_1)(x-x_2) \\
&= \tfrac{1}{3!}\sin(\xi_x)\Pi_{i=0}^{2}(x-x_i)(x-x_0)(x-x_1)(x-x_2) \\
\Longrightarrow \\
|\cos(x) - p_2(x)| &\le \tfrac{1}{3!}|\sin(\xi_x)|\,|(x-x_0)(x-x_1)(x-x_2)| \\
&\le \tfrac{1}{6}\sin(0.4)\times 0.4^3 = 0.0042.
\end{aligned}$$

Problem 3

Prove the identities $\displaystyle\sum_{k=0}^{n}(x-x_k)^m l_{n,k}(x) = 0$ for $m = 1, ..., n$.

Solution: The function $(t-x)^m$, $m = 1, ..., n$, where t is a fixed parameter, is a polynomial of degree at most n and therefore it should coincide with its Lagrange interpolant, i.e. $(t-x)^m = \sum_{j=1}^{n}(t-x_j)^m l_{n,j}(x)$. Now take $t = x$ to get the desired result. Note, that if $m = 0$, then $\sum_{j=1}^{n} l_{n,j}(x) = 1$, which also follows form the construction of $l_{n,j}(x)$.

Another possible solution. For $m = 1, \cdots, n$:

$$\begin{aligned}
\sum_{k=0}^{n}(x-x_k)^m L_{n,k}(x) &= \sum_{k=0}^{n}\sum_{p=0}^{m}\begin{pmatrix} m \\ p \end{pmatrix} x^p(-x_k)^{m-p}L_{n,k}(x) \\
&= \sum_{p=0}^{m}\begin{pmatrix} m \\ p \end{pmatrix} x^p \sum_{k=0}^{n}(-x_k)^{m-p}L_{n,k}(x) \\
&= \sum_{p=0}^{m}\begin{pmatrix} m \\ p \end{pmatrix} x^p(-x)^{m-p} = (x-x)^m = 0
\end{aligned}$$

Problem 4

Let $f(x) = x^n$ and $f[x_0, x_1, ..., x_n]$ be the divided difference of order n using the points $x_0 < x_1 < ... < x_n$. Prove that:

(1) (10 pts) $f[x_0, x_1, ..., x_n] = 1$;
(2) (10 pts) $f[x_0, x_1, ..., x_{n-1}] = x_0 + x_1 + ... + x_{n-1}$.

Solution: One possible solution:
The Newton form of interpolation polynomial is written as:

$$p(x) = \sum_{k=0}^{n} f[x_0, \cdots, x_k]q_k(x), \quad q_k(x) = (x-x_0)(x-x_1)\dots(x-x_{k-1}).$$

Now, $f(x) = x^n$ is a polynomial of degree n, and the $n+1$ distinct interpolation points give a unique interpolation polynomial of degree at most n, so $p(x) = f(x)$. Comparing the coefficients of the x^n term for $p(x)$ and $f(x)$, we have

$$f[x_0, \cdots, x_n] = 1;$$

And Comparing the coefficients of the x^{n-1} term for $p(x)$ and $f(x)$, we have

$$f[x_0, \cdots, x_{n-1}] - f[x_0, \cdots, x_n] \sum_{i=0}^{n-1} x_i = 0$$

i.e.

$$f[x_0, \cdots, x_{n-1}] = \sum_{i=0}^{n-1} x_i = x_0 + x_1 + \cdots + x_{n-1}.$$

Second possible solution:
From our classroom considerations we know that for a given x there is ξ_x such that

$$f[x_0, \cdots, x_{n-1}] = \frac{1}{n!} f^{(n)}(\xi_x).$$

Since the n-th derivative of x^n is equal to $n!$ this gives the desired result.

Problem 5

If $f[x_0, x_1, \ldots, x_n]$ denotes the divided difference of order n prove the **Leibnitz formula**

$$(fg)[x_0, x_1, \ldots, x_n] = \sum_{k=0}^{n} f[x_0, x_1, \ldots, x_k] g[x_k, x_{k+1}, \ldots, x_n].$$

Solution: Assume $x_0, \cdots, x_n$ are distinct points, we prove the formula formally by induction.

For the divided difference of order 0, obivously we have $(fg)[x_0] = f[x_0]g[x_0]$. Suppose the Leibnitz formula is valid for the divided difference of order $n-1$, we show it is valid for order n:

Let $h = fg$. We have the recurrence relationship:

$$h[x_0, x_1, \cdots, x_n] = \frac{h[x_1, \cdots, x_n] - h[x_0, \cdots, x_{n-1}]}{x_n - x_0} \qquad (1)$$

by induction,

$$h[x_1, \cdots, x_n] = \sum_{k=1}^{n} f[x_1, \cdots, x_k] g[x_k, \cdots, x_n]$$

$$h[x_0, \cdots, x_{n-1}] = \sum_{k=0}^{n-1} f[x_0, \cdots, x_k] g[x_k, \cdots, x_{n-1}]$$

so,

$$\begin{aligned}
& h[x_0, x_1, \cdots, x_n](x_n - x_0) \\
= \; & h[x_1, \cdots, x_n] - h[x_0, \cdots, x_{n-1}] \\
= \; & \textstyle\sum_{k=1}^{n} f[x_1, \cdots, x_k]g[x_k, \cdots, x_n] - \sum_{k=0}^{n-1} f[x_0, \cdots, x_k]g[x_k, \cdots, x_{n-1}] \\
= \; & \textstyle\sum_{k=0}^{n-1} f[x_1, \cdots, x_{k+1}]g[x_{k+1}, \cdots, x_n] - \sum_{k=0}^{n-1} f[x_0, \cdots, x_k]g[x_k, \cdots, x_{n-1}] \\
= \; & \textstyle\sum_{k=0}^{n-1} f[x_1, \cdots, x_{k+1}]g[x_{k+1}, \cdots, x_n] - \sum_{k=0}^{n-1} f[x_0, \cdots, x_k]g[x_{k+1}, \cdots, x_n] \\
& \textstyle+\sum_{k=0}^{n-1} f[x_0, \cdots, x_k]g[x_{k+1}, \cdots, x_n] - \sum_{k=0}^{n-1} f[x_0, \cdots, x_k]g[x_k, \cdots, x_{n-1}] \\
= \; & \textstyle\sum_{k=0}^{n-1}(f[x_1, \cdots, x_{k+1}] - f[x_0, \cdots, x_k])g[x_{k+1}, \cdots, x_n] \\
& \textstyle+\sum_{k=0}^{n-1} f[x_0, \cdots, x_k](g[x_{k+1}, \cdots, x_n] - g[x_k, \cdots, x_{n-1}]) \\
(2) \quad = \; & \textstyle\sum_{k=0}^{n-1} f[x_0, x_1, \cdots, x_{k+1}]g[x_{k+1}, \cdots, x_n](x_{k+1} - x_0) \\
& \textstyle+\sum_{k=0}^{n-1} f[x_0, \cdots, x_k]g[x_k, x_{k+1}, \cdots, x_n](x_n - x_k) \\
= \; & \textstyle\sum_{k=1}^{n} f[x_0, x_1, \cdots, x_k]g[x_k, \cdots, x_n](x_k - x_0) \\
& \textstyle+\sum_{k=0}^{n-1} f[x_0, \cdots, x_k]g[x_k, x_{k+1}, \cdots, x_n](x_n - x_k) \\
= \; & f[x_0, \cdots, x_n]g[x_n](x_n - x_0) \\
& \textstyle+\sum_{k=1}^{n-1} f[x_0, \cdots, x_k]g[x_k, \cdots, x_n](x_k - x_0) \\
& \textstyle+\sum_{k=1}^{n-1} f[x_0, \cdots, x_k]g[x_k, \cdots, x_n](x_n - x_k) \\
& +f[x_0]g[x_0, \cdots, x_n](x_n - x_0)
\end{aligned}$$

combining the two sums on the r.h.s. above, we get

(3)

$$\begin{aligned}
& h[x_0, x_1, \cdots, x_n](x_n - x_0) \\
= \; & \textstyle f[x_0, \cdots, x_n]g[x_n](x_n - x_0) + \sum_{k=1}^{n-1} f[x_0, \cdots, x_k]g[x_k, \cdots, x_n](x_k - x_0 + x_n - x_k) \\
& +f[x_0]g[x_0, \cdots, x_n](x_n - x_0) \\
= \; & \textstyle\sum_{k=0}^{n} f[x_0, \cdots, x_k]g[x_k, \cdots, x_n](x_n - x_0)
\end{aligned}$$

dividing $(x_n - x_0)$ on both sides gives:

$$h[x_0, x_1, \cdots, x_n] = \sum_{k=0}^{n} f[x_0, \cdots, x_k]g[x_k, \cdots, x_n]$$

i.e.

$$(fg)[x_0, x_1, \cdots, x_n] = \sum_{k=0}^{n} f[x_0, \cdots, x_k]g[x_k, \cdots, x_n]$$

Problem 6

Let $p(x)$ be the Hermite intepolating polynomial based on $n+1$ distinct points $x_0 < x_1 < \cdots < x_n$ in $[a,b]$. Assume that the data $f_i = f(x_i)$ and $f_i' = f'(x_i)$ is generated by a function $f(x) \in C^{(2n+2)}([a,b])$. Prove that for each $x \in [a,b]$ there is a point ξ_x such that

$$f(x) = p(x) + \frac{1}{(2n+2)!} f^{(2n+2)}(\xi_x)(x-x_0)^2 \dots (x-x_n)^2.$$

Solution: We shall use the technique presented in class. First notice that if x is equal to one of the interpolation points x_j then this reduces to $f(x_j) = p(x_j)$ which is obviously satisfied. Now assume that x in none of the interpolation points x_j and form the function of t

$$\phi(t) = f(t) - p(t) - \lambda w(t), \quad w(t) = (t-x_0)^2(t-x_1)^2 \dots (t-x_n)^2$$

and choose λ in such way that $\phi(x) = 0$, e.g. $\lambda = (f(x) - p(x))/w(x)$. This is possible since x in none of the interpolation points x_j and $w(x) \neq 0$. Obviously $\phi(t)$ vanishes at $x_0, x_1, \dots, x_n, x$, a set of $n+2$ distinct points. By Rolle's Theorem there are $n+1$ points ξ_j each in the intervals defined by the points $x_0, x_1, \dots, x_n, x$ so that $\phi'(\xi_j) = 0$, $j = 1, 2, \dots, n$. Also the derivative ϕ' vanishes at all nodes $x_0, x_1, \dots, x_n$. Thus, alltogether ϕ' vanishes at $2n+2$ points. Applying $2n+1$ times Rolle's theorem we can claim that there is a point ξ_x such that $\phi^{(2n+2)}(\xi_x) = 0$. This leads to $f^{(2n+2)}(\xi_x) - (2n+2)!\lambda = 0$, or $\lambda = f^{(2n+2)}(\xi_x)/(2n+2)!$.

Problem 7

Prove that there is unique polynomial of degree at most $n+2$ that interpolates the data: $f(x_0), f'(x_0), f(x_1), f(x_2), \dots, f(x_{n-1}), f(x_n), f'(x_n)$ at the n distinct points $x_0 < \cdots < x_n$.

Solution: Use the technique presented in class and the reasoning presented in the previous problem.

Problem 8

In your free time and for your amusement(!!!): Show that if

$$\omega(x) = (x-x_0)(x-x_1)...(x-x_n)$$

then:

(1) $\sum_{k=0}^n (x-x_k)^{n+1} l_{n,k}(x) = (-1)^n \omega(x)$
(2) $\sum_{k=0}^n (x-x_k)^{n+2} l_{n,k}(x) = (-1)^n \omega(x) \sum_{k=0}^n (x-x_k)$
(3) $\sum_{k=0}^n l_{n,k}(0) x_k^{n+1} = (-1)^n x_0.x_1...x_n$.

Solution:

(1) Let $f(x) = (t-x)^{n+1}$, then the Lagrange interpolation is

$$p(x) = \sum_{k=0}^n (t-x_k)^{n+1} L_{n,k}(x).$$

From the theorem on polynomial interpolation error, we have,

$$f(x) - p(x) = \frac{1}{(n+1)!} f^{n+1}(\xi_x) \Pi_{i=0}^{n} (x - x_i)$$

i.e.

$$(t-x)^{n+1} - \sum_{k=0}^{n} (t - x_k)^{n+1} L_{n,k}(x) = \frac{1}{(n+1)!} (-1)^{n+1} (n+1)! \omega(x)$$

i.e.

$$(t-x)^{n+1} - \sum_{k=0}^{n} (t - x_k)^{n+1} L_{n,k}(x) = (-1)^{n+1} \omega(x)$$

Let $t = x$, that gives,

$$\sum_{k=0}^{n} (x - x_k)^{n+1} L_{n,k}(x) = (-1)^{n} \omega(x)$$

(2) Let

$$Q(t) = \sum_{k=0}^{n} (x - x_k)^{n+2} L_{n,k}(t) - (x - t)^{n+2} \tag{4}$$

Then,

$$\begin{aligned} Q(x_i) &= \textstyle\sum_{k=0}^{n} (x - x_k)^{n+2} L_{n,k}(x_i) - (x - x_i)^{n+2} \\ &= (x - x_i)^{n+2} - (x - x_i)^{n+2} \\ &= 0 \end{aligned}$$

i.e. $x_i, i = 0, \cdots, n$ are roots of $Q(t)$.
So, $Q(t)$ can be written as

$$Q(t) = (At + B)\omega(t)$$

i.e.

$$\sum_{k=0}^{n} (x - x_k)^{n+2} L_{n,k}(t) - (x - t)^{n+2} = (At + B)\omega(t)$$

where A, B (these might be polynomilas of x) can be determined by comparing the coefficients of the two leading terms of the polynomials on lhs and rhs:
The coefficient of t^{n+2} gives:

$$-(-1)^{n+2} = A$$

i.e. $A = (-1)^{n+1}$.
The coefficient of t^{n+1} gives:

$$-(n+2)x(-1)^{n+1} = B - A \sum_{i=0}^{n} x_i$$

i.e. $B = (n+2)x(-1)^{n} + (-1)^{n+1} \sum_{i=0}^{n} x_i$.

Now, evaluate $Q(t)$ at $t = x$, then

$$
\begin{aligned}
\sum_{k=0}^{n}(x-x_k)^{n+2}L_{n,k}(x)& \\
&= [(-1)^{n+1}x + (n+2)x(-1)^n + (-1)^{n+1}\sum_{i=0}^{n} x_i]\,\omega(t) \\
&= [(n+1)x(-1)^n - (-1)^n \sum_{i=0}^{n} x_i]\,\omega(t) \\
&= (-1)^n \sum_{i=0}^{n}(x - x_i)\,\omega(x)
\end{aligned}
$$

So the claim holds.

(3) Let $f(x) = x^{n+1}$, then the Lagrange interpolation is $p(x) = \sum_{k=0}^{n} L_{n,k}(x)x_k^{n+1}$. From the theorem on polynomial interpolation error, we have,

$$f(x) - p(x) = \frac{1}{(n+1)!} f^{n+1}(\xi_x)\,\omega(x)$$

i.e.

$$x^{n+1} - \sum_{k=0}^{n} L_{n,k}(x)x_k^{n+1} = \frac{1}{(n+1)!}(n+1)!\,\omega(x)$$

Let $x = 0$, then the formula above gives,

$$0 - \sum_{k=0}^{n} L_{n,k}(0)x_k^{n+1} = (-1)^{n+1}\Pi_{i=0}^{n} x_i$$

$$\Longrightarrow \quad \sum_{k=0}^{n} L_{n,k}(0)x_k^{n+1} = (-1)^{n}\Pi_{i=0}^{n} x_i$$

$$i.e. \quad \sum_{k=0}^{n} L_{n,k}(0)x_k^{n+1} = (-1)^{n} x_0 x_1 \cdots x_n.$$