MATH 609-600 Numerical Analysis
Programming assignment #1
Direct methods for linear system by Gaussian elimination

This Programming assignment is worth of 100 pts. Delay is penalized by 5 pts per day.

1. Specifications

(1) Write your own program for Gaussian elimination in any algorithmic language of your choice. The program should use double precision and should be well documented.

(2) Report the computed solution and compare the results obtained by different methods. For the report use the template provided by your TA.

(3) First test your program on the following simple example of a diagonally dominant pentadiagonal matrix: $a_{ii} = 10,\ a_{ij} = 1,\ a_{ij} = 0, |i-j| > 2,\ i,j = 1,...,n,\ n = 10,\ 20$ and $b_1 = b_n = 12, b_2 = b_{n-1} = 13$ and $b_j = 14$, for $j = 3, 4, \ldots, n-2$. (This problem is just to test your program, the solution is $x_1 = x_2 = ... = x_n = 1$; do not report the results).

2. Computational examples

Solve the following linear systems:

(1) (Hilbert matrix): $a_{ij} = 1/(i+j-1),\ x_i = 1$, and $b = Ax,\ i,j = 1,\ldots,n$, $n = 5, 10, 20$. Solve $Ay = b$ and compare your solution y with the exact solution x ($x_i = 1$). Since this matrix is very ill-conditioned, during the execution process you may get almost zero pivot and as a consequence may not be able to solve the system for $n = 20$.

(2) $x_0 = 0,\ k_{i-1}(x_i - x_{i-1}) + k_i(x_i - x_{i+1}) + c_i x_i = b_i,\ i = 1,...,n,\ x_{n+1} = 0$, which after the elimination of x_0 and x_{n+1} is written in the form $Ax = b$ with $x \in R^n$. Solve for the following systems for $n =\ 20,\ 40$:

(1) $k_i = 1$, $i = 0, \ldots, n$, $c_i = 0$ and $b_i = 1$, $i = 1, \ldots, n$;

(2) $k_i = 1$, $i = 0, \ldots, n/2$ and $k_i = 10$, $i = n/2+1, \ldots, n$ and $c_i = 0.1$ and $b_i = 1$, $i = 1, \ldots, n$;

Plot x_i treating x_i as approximations of values of a function $u(x)$, $0 \le x \le 1$, where $x_i \approx u(\frac{i}{n+1})$, i=0,...,n+1.

Remark. The solution x_i is an approximation of $u(t_i)$, $t_i = i/(n+1)$ with $u(t)$ the solution of a boundary value problem $-(K(x)u')' + cu = b$ on the interval $(0,1)$ with boundary conditions $u(0) = u(1) = 0$.

(3) Now the unknowns are given as a two-dimensional array $z_{i,j}, i,j = 0, \ldots, n+1$ that satisfy the system:

$$(4+h^2)z_{i,j} - z_{i-1,j} - z_{i+1,j} - z_{i,j-1} - z_{i,j+1} = h^2,$$
$$z_{0,j} = z_{n+1,j} = z_{i,0} = z_{i,n+1} = 0.$$

Here $h = 1/(n+1)$ so that the system represents a finite difference approximation of the boundary value problem $-\Delta u + u = 1$ in $\Omega = (0,1)\times(0,1)$ and $u = 0$ on the boundary of Ω. Solve for $n = 20,\ 40$. You need to represent this in the standard format $Ax = b$, where A is an $\mathcal{R}^{n^2\times n^2}$ matrix and x is a vector in $\mathcal{R}^{n^2}$ by renaming the unknowns. Use lexicographical enumeration of the unknowns $x_{(i-1)n+j} = u_{i,j}$ so that when $i, j = 1, \dots, n$ we have $k = (i-1)n + j = 1, \dots, n^2$.

Remark. Again, since the solution $z_{i,j}$ of this system represents an approximation of the solution $u(ih, jh)$ of a boundary value problem in Ω it makes sense to plot $x_{i,j}$.

3. Extra Credit (50 pts)

Write your own routine for the Gaussian elimination of a banded matrix with a bandwidth p (n an integer number), such that $n > p \geq 1$. For this write down the matrix as a rectangular $(2p+1)\times n$-matrix, i.e. in $R^{(2p+1)\times n}$, that keeps the highest co-diagonal as first row and the lowest co-diagonal as the last row. In this way the matrix entry of position (i,j), $i, j+1, \dots, n$, $|i-j| \leq p$ in the original matrix will be in $(k, j) = (i - j + p + 1, j)$ position in the new format with $j = 1, \dots, n$ and $k = 1, \dots, 2p + 1$.

The above systems (2) and (3) have banded matrices, so these examples with your program.

Report the timing, if you are able to clock the execution times of these two cases. It might be interesting to see this for very large matrices, say, the second example with $n = 40$.