6 Method of characteristics

Question 80: (a) Show that the PDE $u_y = 0$ in the half plane $\{x > 0\}$ has no solution which is $\mathcal{C}^1$ and satisfies the boundary condition $u(y^2, y) = y$.

Solution: The PDE implies that $u(x, y) = \phi(x)$ where ϕ is any $\mathcal{C}^1$ function. The boundary condition implies $\phi(1) = u(1, -1) = -1$ and $\phi(1) = u(1, 1) = 1$, which is impossible. The reason for this happening is that the characteristics lines ($x = c$) cross the boundary curve (the parabola of equation $x = y^2$) twice.

(b) Find the $\mathcal{C}^1$ function that solves the above PDE in the quadrant $\{x > 0, 0 > y\}$ (beware the sign of y).

Solution: The PDE implies $u(x, y) = \phi(x)$ and the boundary condition implies $\phi(y^2) = u(y^2, y) = y = -|y|$ since y is negative. Then $u(x, y) = \phi(x) = -\sqrt{x}$.

Question 81: Let $\Omega = \{x > 0, y > 0\}$ be the first quadrant of the plane. Let Γ be the line defined by the following parameterization $\Gamma = \{x = s, y = 1/s, s > 0\}$. Solve the following PDE:

$$xu_x + 2yu_y = 0, \quad \text{in } \Omega,$$
$$u(x, y) = x \quad \text{on } \Gamma.$$

Solution: The characteristics are $X(\tau, s) = se^{\tau}$, $Y(\tau, s) = s^{-1}e^{2\tau}$. Upon setting $u(X(\tau, s), Y(\tau, s)) = w(\tau, s)$, we obtain $w(\tau, s) = w(0, s)$. Then the boundary condition implies $w(0, s) = u(s, \frac{1}{s}) = s$. In other words $u(x, y) = (x^2y^{-1})^{1/3}$.

Question 82: (a) Solve the quasi-linear PDE $3u^2u_x + 3u^2u_y = 1$ in the plane by using the method of Lagrange (that is, show that u solves the nonlinear equation $c(a(x, y, u), b(x, y, u)) = 0$ where c is an arbitrary function and a, b are polynomials of degree 3 that you must find.)

Solution: The auxiliary equation is $3z^2\phi_x + 3z^2\phi_y + \phi_z = 0$. Define the plane $\Gamma = \{x = s, y = s', z = 0\}$ and enforce $\phi(x, y, z) = \phi_0(s, s')$ on Γ, where ϕ_0 is an arbitrary $\mathcal{C}^1$ function. The characteristics are $X(\tau, s, s') = \tau^3 + s$ $Y(\tau, s, s') = \tau^3 + s'$, $Z(\tau, s, s') = \tau$. Then $\phi(x, y, z) = \phi_0(s, s')$ where $s = x - z^3$ and $s' = y - z^3$. Then $\phi(x, y, z) = \phi_0(x - z^3, y - z^3)$. Hence, u solves $\phi_0(x - u^3, y - u^3) = 0$.

(b) Find a solution to the above PDE that satisfies the boundary condition $u(x, 2x) = 1$.

Solution: We want $\phi_0(x-1, 2x-1) = 0$. Take $\phi_0(\alpha, \beta) = 2\alpha-\beta+1$. Then $2(x-u^3)-(y-u^3)+1 = 0$, that is $u(x, y) = (1 + 2x - y)^{1/3}$.

Question 83: We want to solve the following PDE:

$$\begin{aligned}
&\partial_t w + 3\partial_x w = 0, \quad x > -t,\ t > 0\\
&w(x,t) = w_\Gamma(x,t),\ \text{for all } (x,t) \in \Gamma \text{ where}\\
&\Gamma = \{(x,t) \in \mathbb{R}^2 \text{ s.t. } x = -t,\ x < 0\} \cup \{(x,t) \in \mathbb{R}^2 \text{ s.t. } t = 0,\ x \geq 0\}\\
&\text{and } w_\Gamma \text{ is a given function.}
\end{aligned}$$

(a) Draw a picture of the domain Ω where the PDE must be solved, of the boundary Γ, and of the characteristics.

Solution:

(b) Define a one-to-one parametric representation of the boundary Γ.

Solution: For negative s we set $x_\Gamma(s) = s$ and $t_\Gamma(s) = -s$; clearly we have $x_\Gamma(s) = -t_\Gamma(s)$ for all $s < 0$. For positive s we set $x_\Gamma(s) = s$ and $t_\Gamma(s) = 0$. The map $\mathbb{R} \in s \mapsto (x_\Gamma(s), t_\Gamma(s)) \in \Gamma$ is one-t-one.

(c) Give a parametric representation of the characteristics associated with the PDE.

Solution: (i) We use t and s to parameterize the characteristics. The characteristics are defined by

$$\partial_t X(t,s) = 3, \quad \text{with} \quad x(t_)\Gamma(s), s) = x_\Gamma(s).$$

This yields the following parametric representation of the characteristics

$$X(t,s) = 3(t - t_\Gamma(s)) + x_\Gamma(s),$$

where $t \geq 0$ and $s \in (-\infty, +\infty)$.

(d) Give an implicit parametric representation of the solution to the PDE.

Solution: (i) Now we set $\phi(t,s) = w(X(t,s), t(t,s))$ and we insert this ansatz in the equation. This gives $\frac{d\phi}{dt}(t,s) = 0$, i.e., $\phi(t,s)$ does not depend on t. In other words

$$w(X(t,s), t(t,s)) = \phi(t,s) = \phi(0,s) = w(x(0,s), t(0,s)) = w_\Gamma(x_\Gamma(s), t_\Gamma(s))$$

A parametric representation of the solution is given by

$$\begin{aligned}
&X(t,s) = 3(t - t_\Gamma(s)) + x_\Gamma(s),\\
&w(X(t,s), t(t,s)) = w_\Gamma(x_\Gamma(s), t_\Gamma(s)).
\end{aligned}$$

(e) Give an explicit representation of the solution.

Solution: (i) We have to find the inverse map $(x,t) \mapsto (t,s)$. Clearly $x - 3t = x_\Gamma(s) - 3t_\Gamma(s)$. Then, there are two cases depending on the sign of s.
case 1: If $s < 0$, then $x_\Gamma(s) = s$ and $t_\Gamma(s) = -s$. That means $x - 3t = 4s$, which in turns implies $s = \frac{1}{4}(x - 3t)$. Then

$$w(x,t) = w_\Gamma(\tfrac{1}{4}(x - 3t), -\tfrac{1}{4}(x - 3t)), \quad \text{if } x - 3t < 0.$$

case 2: If $s \geq 0$, then $x_\Gamma(s) = s$ and $t_\Gamma(s) = 0$. That means $x - 3t = s$. Then

$$w(x,t) = w_\Gamma(x - 3t, 0), \quad \text{if } x - 3t \geq 0.$$

Note that the explicit representation of the solution does not depend on the choice of the parameterization.

Question 84: Solve the following PDE by the method of characteristics:

$$\begin{aligned}
&\partial_t w + 3\partial_x w = 0, \quad x > 0,\ t > 0\\
&w(x,0) = f(x), \quad x > 0, \quad \text{and} \quad w(0,t) = h(t), \quad t > 0.
\end{aligned}$$

Solution: First we parameterize the boundary of Ω by setting $\Gamma = \{x = x_\Gamma(s), t = t_\Gamma(s);\ s \in \mathbb{R}\}$ with

$$x_\Gamma(s) = \begin{cases} 0 & \text{if } s < 0, \\ s, & \text{if } s \geq 0. \end{cases} \quad \text{and} \quad t_\Gamma(s) = \begin{cases} -s & \text{if } s < 0, \\ 0, & \text{if } s \geq 0. \end{cases}$$

The we define the characteristics by

$$\partial_t X(s,t) = 3, \quad \text{with} \quad X(s, t_\Gamma(s)) = x_\Gamma(s).$$

The general solution is $X(s,t) = 3(t - t_\Gamma(s)) + x_\Gamma(s)$. Now we make the change of variable $\phi(s,t) = w(X(s,t),t)$ and we compute $\partial_t\phi(s,t)$,

$$\partial_t\phi(s,t) = \partial_t w(X(s,t),t) + \partial_x w(X(s,t),t)\partial_t X(s,t) = \partial_t w(X(s,t),t) + 3\partial_x w(X(s,t),t) = 0.$$

This means that $\phi(s,t) = \phi(s, t_\Gamma(s))$. In other words

$$w(X(s,t),t) = w(X(s,t_\Gamma(s)), t_\Gamma(s)) = w(x_\Gamma(s), t_\Gamma(s)).$$

<u>Case 1:</u> If $s < 0$, then $X(s,t) = 3(t - t_\Gamma(s))$. This implies $t_\Gamma(s) = t - X/3$. The condition $s < 0$ and the definition $t_\Gamma(s) = -s$ imply $t - X/3 \geq 0$. Moreover we have

$$w(X,t) = w(0, t_\Gamma(s)) = h(t_\Gamma(s)).$$

In conlusion

$$w(X,t) = h(t - X/3), \quad \text{if} \quad 3t > X.$$

<u>Case 2:</u> If $s \geq 0$, then $X(s,t) = 3t + x_\Gamma(s)$. This implies $x_\Gamma(s) = X - 3t$. The condition $s \geq 0$ and the definition $x_\Gamma(s) = s$ imply $X - 3t \geq 0$. Moreover we have

$$w(X,t) = w(x_\Gamma(s), 0) = f(x_\Gamma(s)).$$

In conlusion

$$w(X,t) = f(X - 3t), \quad \text{if} \quad X \geq 3t.$$

Question 85: Let $\Omega = \{(x,t) \in \mathbb{R}^2;\ x + 2t \geq 0\}$. Solve the following PDE in explicit form with the method of characteristics:

$$\partial_t u(x,t) + 3\partial_x u(x,t) = u(x,t), \quad \text{in } \Omega, \quad \text{and} \quad u(x,t) = 1 + \sin(x), \text{ if } x + 2t = 0.$$

Solution: (i) First we parameterize the boundary of Ω by setting $\Gamma = \{x = x_\Gamma(s), t = t_\Gamma(s);\ s \in \mathbb{R}\}$ with $x_\Gamma(s) = -2s$ and $t_\Gamma(s) = s$. This choice implies

$$u(x_\Gamma(s), t_\Gamma(s)) := u_\Gamma(s) := 1 + \sin(-2s).$$

(ii) We compute the characteristics

$$\partial_t X(t,s) = 3, \quad X(t_\Gamma(s), s) = x_\Gamma(s).$$

The solution is $X(t,s) = 3(t - t_\Gamma(s)) + x_\Gamma(s)$.

(iii) Set $\Phi(t,s) := u(X(t,s),t)$ and compute $\partial_t\Phi(t,s)$. This gives

$$\begin{aligned}\partial_t\Phi(t,s) &= \partial_t u(X(t,s),t) + \partial_x u(X(t,s),t)\partial_t X(t,s) \\ &= \partial_t u(X(t,s),t) + 3\partial_x u(X(t,s),t) = u(X(t,s),t) = \Phi(t,s).\end{aligned}$$

The solution is $\Phi(t,s) = \Phi(t_\Gamma(s), s)e^{t - t_\Gamma(s)}$.

(iv) The implicit representation of the solution is

$$X(t,s) = 3(t - t_\Gamma(s)) + x_\Gamma(s) \quad u(X(t,s)) = u_\Gamma(s)e^{t - t_\Gamma(s)}.$$

(v) The explicit representation is obtained by using the definitions of $-t_\Gamma(s)$, $x_\Gamma(s)$ and $u_\Gamma(s)$.

$$X(s,t) = 3(t-s) - 2s = 3t - 5s,$$

which gives

$$s = \frac{1}{5}(3t - X).$$

The solution is

$$\begin{aligned} u(x,t) &= (1+\sin(\tfrac{2}{5}(x-3t)))e^{t-\frac{1}{5}(3t-x)} \\ &= (1+\sin(\tfrac{2(x-3t)}{5}))e^{\frac{x+2t}{5}}. \end{aligned}$$

Question 86: Let $\Omega = \{(x,t) \in \mathbb{R}^2;\ x \geq 0,\ t \geq 0\}$. Solve the following PDE in explicit form

$$\partial_t u(x,t) + t\partial_x u(x,t) = 2u(x,t), \quad \text{in } \Omega, \quad \text{and} \quad u(0,t) = t,\ u(x,0) = x.$$

Solution: (i) First we parameterize the boundary of Ω by setting $\Gamma = \{x = x_\Gamma(s), t = t_\Gamma(s);\ s \in \mathbb{R}\}$ with $x_\Gamma(s) = s$ and $t_\Gamma(s) = 0$ if $s > 0$ and $x_\Gamma(s) = 0$ and $t_\Gamma(s) = -s$ if $s \leq 0$. This choice implies

$$u(x_\Gamma(s), t_\Gamma(s)) := u_\Gamma(s) := \begin{cases} s & \text{if } s > 0 \\ -s & \text{if } s \leq 0 \end{cases}.$$

(ii) We compute the characteristics

$$\partial_t X(t,s) = t, \quad X(t_\Gamma(s), s) = x_\Gamma(s).$$

The solution is $X(t,s) = \frac{1}{2}t^2 - \frac{1}{2}t_\Gamma^2(s) + x_\Gamma(s)$.

(iii) Set $\Phi(t,s) := u(X(t,s),t)$ and compute $\partial_t \Phi(t,s)$. This gives

$$\begin{aligned} \partial_t \Phi(t,s) &= \partial_t u(X(t,s),t) + \partial_x u(X(t,s),t)\partial_t X(t,s) \\ &= \partial_t u(X(t,s),t) + t\partial_x u(X(t,s),t) = 2u(X(t,s),t) = 2\Phi(t,s). \end{aligned}$$

The solution is $\Phi(t,s) = \Phi(t_\Gamma(s), s)e^{2(t-t_\Gamma(s))}$.

(iv) The implicit representation of the solution is

$$X(t,s) = \frac{1}{2}t^2 - \frac{1}{2}t_\Gamma^2(s) + x_\Gamma(s), \quad u(X(t,s)) = u_\Gamma(s)e^{2(t-t_\Gamma(s))}, \quad u_\Gamma(s) = \begin{cases} s & \text{if } s > 0 \\ -s & \text{if } s \leq 0 \end{cases}.$$

(v) We distinguish two cases to get the explicit form of the solution:
Case 1: Assume $s > 0$, then $t_\Gamma(s) = 0$ and $x_\Gamma(s) = s$. This implies $X(t,s) = \frac{1}{2}t^2 + s$, meaning $s = X - \frac{1}{2}t^2$. The solution is

$$u(x,t) = (x - \frac{1}{2}t^2)e^{2t}, \quad \text{if} \quad x > \frac{1}{2}t^2.$$

Case 2: Assume $s \leq 0$, then $t_\Gamma(s) = -s$ and $x_\Gamma(s) = 0$. This implies $X(t,s) = \frac{1}{2}t^2 - \frac{1}{2}s^2$, meaning $s = -\sqrt{t^2 - 2X}$. The solution is

$$u(x,t) = \sqrt{t^2 - 2x}\, e^{2(t - \sqrt{t^2-2x})}, \quad \text{if} \quad x \leq \frac{1}{2}t^2.$$

Question 87: Let $\Omega = \{(t,x) \in \mathbb{R}^2 :\ t > 0,\ x \geq t\}$. Let Γ be defined by the following parameterization $\Gamma = \{x = x_\Gamma(s), t = t_\Gamma(s), s \in \mathbb{R}\}$, with $x_\Gamma(s) = -s$ and $t_\Gamma(s) = -s$ if $s \leq 0$,

$x_\Gamma(s) = s$ and $t_\Gamma(s) = 0$ if $s \geq 0$. Solve the following PDE (give the implicit and explicit representations):

$$u_t + 3u_x + 2u = 0, \quad \text{in } \Omega, \qquad u(x,t) = u_\Gamma(x,t) := \begin{cases} 1 & \text{if } t = 0 \\ 2 & \text{if } x = t \end{cases} \quad \text{for all } (x,t) \text{ in } \Gamma.$$

Solution: We define the characteristics by

$$\frac{dx(t,s)}{dt} = 3, \quad x(t_\Gamma(s),s) = x_\Gamma(s).$$

This gives $x(t,s) = x_\Gamma(s) + 3(t - t_\Gamma(s))$. Upon setting $\phi(t,s) = u(x(t,s),t)$, we observe that $\partial_t\phi(t,s) + 2\phi(t,s) = 0$, which means

$$\phi(t,s) = ce^{-2t}.$$

The initial condition implies $\phi(t_\Gamma(s),s) = u_\Gamma(x_\Gamma(s),t_\Gamma(s))$; as a result $c = u_\Gamma(x_\Gamma(s),t_\Gamma(s))e^{2t_\Gamma(s)}$.

$$\phi(t,s) = u_\Gamma(x_\Gamma(s),t_\Gamma(s))e^{2(t_\Gamma(s)-t)}.$$

The implicit representation of the solution is

$$u(x(t,s),t) = u_\Gamma(x_\Gamma(s),t_\Gamma(s))e^{2(t_\Gamma(s)-t)}, \qquad x(t,s) = x_\Gamma(s) + 3(t - t_\Gamma(s)).$$

Now we give the explicit representation.
Case 1: If $s \leq 0$, $x_\Gamma(s) = -s$, $t_\Gamma(s) = -s$, and $u_\Gamma(x_\Gamma(s),t_\Gamma(s)) = 2$. This means $x(t,s) = -s + 3(t+s)$ and we obtain $s = \frac{1}{2}(x - 3t)$, which means

$$u(x,t) = 2e^{-2(\frac{1}{2}(x-3t)-t)} = 2e^{t-x}, \qquad \text{if } x - 3t < 0.$$

Case 2: If $s \geq 0$, $x_\Gamma(s) = s$, $t_\Gamma(s) = 0$, and $u_\Gamma(x_\Gamma(s),t_\Gamma(s)) = 1$. This means $x(t,s) = s + 3t$ and we obtain $s = x - 3t$, which means

$$u(x,t) = e^{-2t}, \qquad \text{if } x - 3t > 0.$$

Question 88: Let $\Omega = \{(t,x) \in \mathbb{R}^2 : \ t > 0, \ x \geq -\sqrt{t}\}$. Let Γ be defined by the following parameterization $\Gamma = \{x = x_\Gamma(s), t = t_\Gamma(s), s \in \mathbb{R}\}$, with $x_\Gamma(s) = s$ and $t_\Gamma(s) = s^2$ if $s \leq 0$, $x_\Gamma(s) = s$ and $t_\Gamma(s) = 0$ if $s \geq 0$. Solve the following PDE (give the implicit and explicit representations):

$$u_t + 2u_x + 3u = 0, \quad \text{in } \Omega, \qquad \text{and} \quad u(x_\Gamma(s),t_\Gamma(s)) := e^{-t_\Gamma(s)-x_\Gamma(s)}, \quad \forall s \in (-\infty,+\infty).$$

Solution: We define the characteristics by

$$\frac{dX(t,s)}{dt} = 2, \quad X(t_\Gamma(s),s) = x_\Gamma(s).$$

This gives $X(t,s) = x_\Gamma(s) + 2(t - t_\Gamma(s))$. Upon setting $\phi(t,s) = u(X(t,s),t)$, we observe that $\partial_t\phi(t,s) + 3\phi(t,s) = 0$, which means

$$\phi(t,s) = ce^{-3t}.$$

The initial condition implies $\phi(t_\Gamma(s),s) = u(x_\Gamma(s),t_\Gamma(s)) = e^{-t_\Gamma(s)-x_\Gamma(s)} = ce^{-3t_\Gamma(s)}$; as a result $c = e^{2t_\Gamma(s)-x_\Gamma(s)}$ and

$$\phi(t,s) = e^{2t_\Gamma(s)-x_\Gamma(s)-3t}.$$

The implicit representation of the solution is

$$u(X(t,s),t) = e^{2t_\Gamma(s)-x_\Gamma(s)-3t}, \qquad X(t,s) = x_\Gamma(s) + 2(t - t_\Gamma(s)).$$

Now we give the explicit representation.
We observe the following:

$$2t_\Gamma(s) - x_\Gamma(s) = 2t - X(t,s),$$

which gives

$$u(X(t,s),t) = e^{2t-X(t,s)-3t} = e^{-X(t,s)-t}.$$

In conclusion, the explicit representation of the solution to the problem is the following:

$$u(x,t) = e^{-x-t}.$$

Question 89: Let $\Omega = \{(t,x) \in \mathbb{R}^2 : \ t > 0, \ x \geq -t\}$. Let Γ be defined by the following parameterization $\Gamma = \{x = x_\Gamma(s), t = t_\Gamma(s), s \in \mathbb{R}\}$, with $x_\Gamma(s) = s$ and $t_\Gamma(s) = -s$ if $s \leq 0$, $x_\Gamma(s) = s$ and $t_\Gamma(s) = 0$ if $s \geq 0$. Solve the following PDE (give the implicit and explicit representations):

$$u_t + 2u_x + u = 0, \quad \text{in } \Omega, \qquad u(x,t) = u_\Gamma(x,t) := \begin{cases} 1 & \text{if } x > 0 \\ 2 & \text{if } x < 0 \end{cases} \quad \text{for all } (x,t) \text{ in } \Gamma.$$

Solution: We define the characteristics by

$$\frac{dx(t,s)}{dt} = 2, \quad x(t_\Gamma(s), s) = x_\Gamma(s).$$

This gives $x(t,s) = x_\Gamma(s) + 2(t - t_\Gamma(s))$. Upon setting $\phi(t,s) = u(x(t,s),t)$, we observe that $\partial_t \phi(t,s) + \phi(t,s) = 0$, which means

$$\phi(t,s) = ce^{-t}.$$

The initial condition implies $\phi(t_\Gamma(s), s) = u_\Gamma(x_\gamma(s), t_\Gamma(s))$; as a result $c = u_\Gamma(x_\Gamma(s), t_\Gamma(s))e^{t_\Gamma(s)}$.

$$\phi(t,s) = u_\Gamma(x_\Gamma(s), t_\Gamma(s))e^{t_\Gamma(s)-t}.$$

The implicit representation of the solution is

$$u(x(t,s),t) = u_\Gamma(x_\Gamma(s), t_\Gamma(s))e^{t_\Gamma(s)-t}, \qquad x(t,s) = x_\Gamma(s) + 2(t - t_\Gamma(s)).$$

Now we give the explicit representation.
Case 1: If $s \leq 0$, $x_\Gamma(s) = s$, $t_\Gamma(s) = -s$, and $u_\Gamma(x_\Gamma(s), t_\Gamma(s)) = 2$. This means $x(t,s) = s+2(t+s)$ and we obtain $s = \frac{1}{3}(x - 2t)$, which means

$$u(x,t) = 2e^{-\frac{1}{3}(x-2t)-t}, \qquad \text{if } x - 2t < 0.$$

Case 2: If $s \geq 0$, $x_\Gamma(s) = s$, $t_\Gamma(s) = 0$, and $u_\Gamma(x_\Gamma(s), t_\Gamma(s)) = 1$. This means $x(t,s) = s + 2t$ and we obtain $s = x - 2t$, which means

$$u(x,t) = e^{-t}, \qquad \text{if } x - 2t > 0.$$

Question 90: Let $\Omega = \{(x,t) \in \mathbb{R}^2 \mid t > 0, x \geq \frac{1}{t}\}$. Solve the following PDE in explicit form with the method of characteristics: (Solution: $u(x,t) = (2 + \cos(s))e^{\frac{1}{s}-t}$ with $s = \frac{1}{2}[(x-2t) + \sqrt{(x-2t)^2 + 8}])$

$$\partial_t u(x,t) + 2\partial_x u(x,t) = -u(x,t), \quad \text{in } \Omega, \quad \text{and} \quad u(x,t) = 2 + \cos(x), \text{ if } x = 1/t.$$

Solution: (i) First we parameterize the boundary of Ω by setting $\Gamma = \{x = x_\Gamma(s), t = t_\Gamma(s); \ s \in \mathbb{R}\}$ with $x_\Gamma(s) = s$ and $t_\Gamma(s) = \frac{1}{s}$. This choice implies

$$u(x_\Gamma(s), t_\Gamma(s)) := u_\Gamma(s) := 2 + \cos(s).$$

(ii) We compute the characteristics

$$\partial_t X(t,s) = 2, \quad X(t_\Gamma(s),s) = x_\Gamma(s).$$

The solution is $X(t,s) = 2(t - t_\Gamma(s)) + x_\Gamma(s)$.

(iii) Set $\Phi(t,s) := u(X(t,s),t)$ and compute $\partial_t \Phi(t,s)$. This gives

$$\begin{aligned}\partial_t \Phi(t,s) &= \partial_t u(X(t,s),t) + \partial_x u(X(t,s),t)\partial_t X(t,s) \\ &= \partial_t u(X(t,s),t) + 2\partial_x u(X(t,s),t) = u(X(t,s),t) = -\Phi(t,s).\end{aligned}$$

The solution is $\Phi(t,s) = \Phi(t_\Gamma(s),s)e^{-t+t_\Gamma(s)}$.

(iv) The implicit representation of the solution is

$$X(t,s) = 2(t - t_\Gamma(s)) + x_\Gamma(s), \qquad u(X(t,s)) = u_\Gamma(s)e^{-t+t_\Gamma(s)}.$$

(v) The explicit representation is obtained by using the definitions of $-t_\Gamma(s)$, $x_\Gamma(s)$ and $u_\Gamma(s)$.

$$X(s,t) = 2(t - \frac{1}{s}) + s = 2t - \frac{2}{s} + s$$

which gives the equation

$$s^2 - s(X - 2t) - 2 = 0$$

The solutions are $s_\pm = \frac{1}{2}\left((X-2t) \pm \sqrt{(X-2t)^2+8}\right)$. The only legitimate solution is the positive one:

$$s = \frac{1}{2}\left((X-2t) + \sqrt{(X-2t)^2+8}\right)$$

The solution is

$$\begin{aligned}&u(x,t) = (2+\cos(s))e^{\frac{1}{s}-t} \\ &\text{with } s = \frac{1}{2}((x-2t) + \sqrt{(x-2t)^2+8})\end{aligned}$$