7 Conservation equations

The implicit representation of the solution to the equation $\partial_t v + \partial_x q(v) = 0$, $v(x,0) = v_0(x)$, is

$$X(s,t) = q'(v_0(s))t + s; \quad v(X(s,t),t) = v_0(s). \tag{12}$$

Question 91: Consider the following conservation equation

$$\partial_t\rho + \partial_x(q(\rho)) = 0, \quad x \in (-\infty,+\infty),\ t > 0, \qquad \rho(x,0) = \rho_0(x) := \begin{cases} \frac{1}{6} & \text{if } x < 0, \\ \frac{1}{3} & \text{if } x > 0, \end{cases}$$

where $q(\rho) = \rho(2 - 3\rho)$ (and $\rho(x,t)$ is the conserved quantity). Solve this problem using the method of characteristics. Do we have a shock or an expansion wave here?

Solution: The characteristics are defined by

$$\frac{dX(s,t)}{dt} = q'(\rho) = 2(1 - 3\rho(X(s,t),t)), \quad X(s,t) = s.$$

Set $\phi(s,t) = \rho(X(s,t),t)$, then we obtain that ϕ is constant, i.e., ρ is constant along the characteristics: $\rho(X(s,t),t) = \rho(s,0) = \rho_0(s)$. As a result we can integrate the equation defining the characteristics and we obtain $X(t) = 2(1-3\rho_0(s))t+s$. We then have two cases depending whether s is positive or negative.

1. $s < 0$, then $\rho_0(s) = \frac{1}{6}$ and $X(s,t) = t + s$. This means
$$\rho(x,t) = \frac{1}{6} \quad \text{if} \quad x < t.$$
2. $s > 0$, then $\rho_0(s) = \frac{1}{3}$ and $X(s,t) = s$. This means
$$\rho(x,t) = \frac{1}{3} \quad \text{if} \quad x > 0.$$

We see that the characteristics cross in the region $\{t > x > 0\}$. This implies that there is a shock. The Rankin-Hugoniot relation gives the speed of this shock:

$$s = \frac{q^+ - q^-}{\rho^+ - \rho^-} = \frac{\frac{1}{6}\frac{3}{2} - \frac{1}{3}}{\frac{1}{6} - \frac{1}{3}} = \frac{1}{12}6 = \frac{1}{2}.$$

In conclusion

$$\rho = \frac{1}{6}, \quad x < \frac{t}{2},$$
$$\rho = \frac{1}{3}, \quad x > \frac{t}{2}.$$

Question 92: Consider the following conservation equation

$$\partial_t\rho + \partial_x(q(\rho)) = 0, \quad x \in (-\infty,+\infty),\ t > 0, \qquad \rho(x,0) = \rho_0(x) := \begin{cases} 3 & \text{if } x < 0, \\ 1 & \text{if } x > 0, \end{cases}$$

where $q(\rho) = \rho(2 + \rho)$ (and $\rho(x,t)$ is the conserved quantity). Solve this problem using the method of characteristics. Do we have a shock or an expansion wave here?

Solution: The characteristics are defined by

$$\frac{dX(t)}{dt} = q'(\rho) = 2(1 + \rho(x(t),t)), \quad X(0) = X_0.$$

Set $\phi(t) = \rho(X(t),t)$, then we obtain that ϕ is constant, i.e., ρ is constant along the characteristics: $\rho(X(t),t) = \rho(X_0,0) = \rho_0(X_0)$. As a result we can integrate the equation defining the characteristics and we obtain $X(t) = 2(1 + \rho_0(X_0))t + X_0$. We then have two cases depending whether X_0 is positive or negative.

1. $X_0 < 0$, then $\rho_0(X_0) = 3$ and $X(t) = 2(1+3)t + X_0 = 8t + X_0$. This means
$$\rho(x,t) = 3 \quad \text{if} \quad x < 8t.$$
2. $X_0 > 0$, then $\rho_0(X_0) = 1$ and $X(t) = 2(1+1)t + X_0 = 4t + X_0$. This means
$$\rho(x,t) = 1 \quad \text{if} \quad x > 4t.$$

We see that the characteristics cross in the region $\{8t > x > 4t\}$. This implies that there is a shock. The Rankin-Hugoniot relation gives the speed of this shock:

$$\frac{dx_s(t)}{dt} = \frac{q^+ - q^-}{\rho^+ - \rho^-} = \frac{15-3}{3-1} = 6, \qquad x_s(0) = 0.$$

In conclusion, $x_s(t) = 6t$ and

$$\rho = 3, \quad x < x_s(t) = 6t,$$
$$\rho = 1, \quad x > x_s(t) = 6t.$$

Question 93: Solve the conservation equation $\partial_t \rho + \partial_x q(\rho) = 0$, $x \in (\infty, +\infty)$, $t > 0$ with flux $q(\rho) = \rho^2 + \rho$, and with the initial condition $\rho(x,0) = -1$, if $x < 0$, $\rho(x,0) = 1$, if $x > 0$. Do we have a shock or an expansion wave here?

Solution: The solution is given by the implicit representation

$$\rho(X(s,t),t) = \rho_0(s), \quad X(s,t) = s + (2\rho_0(s) + 1)t.$$

Case 1: $s < 0$. Then $\rho_0(s) = -1$ and $X(s,t) = s + (-2+1)t$. This means $s = X + t$. The solution is

$$\rho(x,t) = -1, \quad \text{if } x < t.$$

Case 2: $s < 0$. Then $\rho_0(s) = 1$ and $X(s,t) = s + (2+1)t$. This means $s = X - 3t$. The solution is

$$\rho(x,t) = 1, \quad \text{if } 3t < x.$$

We have a expansion wave. We need to consider the case $\rho_0 \in [-1,1]$ at $s = 0$.
Case 3: $s = 0$ and $\rho_0 \in [-1,1]$. Then $X(s,t) = s + (2\rho_0 + 1)t = (2\rho_0 + 1)t$. This means $\rho_0 = (X/t - 1)2$. In conclusion

$$\rho(x,t) = \frac{1}{2}\left(\frac{x}{t} - 1\right), \quad \text{if } -t < x < 3t.$$

Question 94: Solve the conservation equation $\partial_t \rho + \partial_x q(\rho) = 0$, $x \in (\infty, +\infty)$, $t > 0$ with flux $q(\rho) = \rho^4 + 2\rho$, and with the initial condition $\rho(x,0) = 1$, if $x < 0$, $\rho(x,0) = -1$, if $x > 0$. Do we have a shock or an expansion wave here?

Solution: The solution is given by the implicit representation

$$\rho(X(s,t),t) = \rho_0(s), \quad X(s,t) = s + (4\rho_0(s)^3 + 2)t.$$

We then have two cases depending whether s is positive or negative.
Case 1: $s < 0$, then $\rho_0(s) = 1$ and $X(s,t) = (4+2)t + s = 6t + s$. This means

$$\rho(x,t) = 1 \quad \text{if} \quad x < 6t.$$

Case 2: $s > 0$, then $\rho_0(s) = -1$ and $X(s,t) = (-4+2)t + s = -2t + s$. This means

$$\rho(x,t) = -1 \quad \text{if} \quad x > -2t.$$

We see that the characteristics cross in the region $\{6t > x > -2t\}$. This implies that there is a shock. The Rankin-Hugoniot relation gives the speed of this shock with $\rho^- = 1$ and $\rho^+ = -1$:

$$\frac{dx_s(t)}{dt} = \frac{q^+ - q^-}{\rho^+ - \rho^-} = \frac{-1-3}{-1-1} = 2, \qquad x_s(0) = 0.$$

In conclusion the location of the shock is $x_s(t) = 2t$ and the solution is as follows:

$$\rho = 1, \quad x < x_s(t) = 2t,$$
$$\rho = -1, \quad x > x_s(t) = 2t.$$

Question 95: Consider the following conservation equation

$$\partial_t \rho + \partial_x(q(\rho)) = 0, \quad x \in (-\infty, +\infty),\ t > 0, \qquad \rho(x,0) = \rho_0(x) := \begin{cases} \frac{1}{2} & \text{if } x < 0, \\ 1 & \text{if } x > 0, \end{cases}$$

where $q(\rho) = \rho(2-\rho)$ (and $\rho(x,t)$ is the conserved quantity). Solve this problem using the method of characteristics. Do we have a shock or an expansion wave here?

Solution: The characteristics are defined by

$$\frac{dX(t)}{dt} = q'(\rho) = 2(1 - \rho(x(t),t)), \quad X(0) = X_0.$$

Set $\phi(t) = \rho(X(t),t)$, then we obtain that ϕ is constant, i.e., ρ is constant along the characteristics: $\rho(X(t),t) = \rho(X_0,0) = \rho_0(X_0)$. As a result we can integrate the equation defining the characteristics and we obtain $X(t) = 2(1-\rho_0(X_0))t + X_0$. We then have two cases depending whether X_0 is positive or negative.

1. $X_0 < 0$, then $\rho_0(X_0) = \frac{1}{2}$ and $X(t) = t + X_0$. This means

$$\rho(x,t) = \frac{1}{2} \quad \text{if} \quad x < t.$$

2. $X_0 > 0$, then $\rho_0(X_0) = 1$ and $X(t) = X_0$. This means

$$\rho(x,t) = 1 \quad \text{if} \quad x > 0.$$

We see that the characteristics cross in the region $\{t > x > 0\}$. This implies that there is a shock. The Rankin-Hugoniot relation gives the speed of this shock:

$$s = \frac{q^+ - q^-}{\rho^+ - \rho^-} = \frac{\frac{3}{4} - 1}{\frac{1}{2} - 1} = \frac{1}{2}.$$

In conclusion

$$\rho = \frac{1}{2}, \quad x < \frac{t}{2},$$
$$\rho = 1, \quad x > \frac{t}{2}.$$

Question 96: Consider the following conservation equation

$$\partial_t \rho + \partial_x(q(\rho)) = 0, \quad x \in (-\infty, +\infty),\ t > 0, \qquad \rho(x,0) = \rho_0(x) := \begin{cases} 2 & \text{if } x < 0, \\ 1 & \text{if } x > 0, \end{cases}$$

where $q(\rho) = \rho(2-\rho)$ (and $\rho(x,t)$ is the conserved quantity). Solve this problem using the method of characteristics. Do we have a shock or an expansion wave here?

Solution: The characteristics are defined by

$$\frac{dX(t,x_0)}{dt} = q'(\rho) = 2(1-\rho(X(t,x_0),t)), \quad X(0,x_0) = x_0.$$

Set $\phi(t) = \rho(X(t,x_0),t)$ and insert in the equation. We obtain that $\partial_t\phi(t,x_0) = 0$; meaning that $\phi(t,x_0) = \phi(0,x_0)$, i.e., ρ is constant along the characteristics: $\rho(X(t,x_0),t) = \rho(x_0,0) = \rho_0(x_0)$. As a result we can integrate the equation defining the characteristics and we obtain $X(t,x_0) = 2(1-\rho_0(x_0))t + x_0$. The implicit representation of the solution is

$$X(t,x_0) = 2(1-\rho_0(x_0))t + x_0; \quad \rho(X(t,x_0),t) = \rho_0(x_0)$$

We then have two cases depending whether x_0 is positive or negative.
Case 1: $x_0 < 0$, then $\rho_0(x_0) = 2$ and $X(t,x_0) = 2(1-2)t + x_0 = -2t + x_0$. This means $x_0 = X(t,x_0) + 2t$ and

$$\rho(x,t) = 2 \quad \text{if} \quad x < -2t.$$

Case 2: $x_0 > 0$, then $\rho_0(x_0) = 1$ and $X(t,x_0) = 2(1-1)t + x_0 = x_0$. This means $x_0 = X(t,x_0)$ and

$$\rho(x,t) = 1 \quad \text{if} \quad 0 < x.$$

We see that there is a gap in the region $\{-2t < x < 0\}$. This implies that there is an expansion wave. We have to consider a third case $x_0 = 0$ and $\rho_0 \in (1,2)$.
Case 3: $x_0 = 0$, then $X(t,x_0) = 2(1-\rho_0)t$, i.e., $\rho_0 = 1 - \frac{X(t,x_0)}{2t}$. This means that

$$\rho(x,t) = 1 - \frac{x}{2t}, \quad \text{if} \quad -2t < x < 0.$$

Question 97: Assume $u_1 > u_2 \geq u_3 \geq 0$ and consider the following conservation equation

$$\partial_t u + u\partial_x u = 0, \quad x \in (-\infty, +\infty),\ t > 0, \qquad u(x,0) = u_0(x) := \begin{cases} 0 & \text{if } x \leq 0, \\ u_1 x & \text{if } 0 < x \leq 1, \\ u_1 & \text{if } 1 < x \leq 2, \\ u_2 & \text{if } 2 < x \leq 3, \\ u_3 & \text{if } 3 \leq x. \end{cases}$$

(i) Assume $u_2 = u_3$. Solve until the expansion catches up the shock. When does it happen?

Solution: The characteristics are defined by

$$\frac{dX(t,s)}{dt} = u(X(t,s),t), \qquad X(0,s) = s.$$

From class we know that $u(X(t,s),t)$ does not depend on time, that is to say

$$X(t,s) = u(X(0,s),0)t + s = u(s,0)t + x_0 = u_0(s)t + s.$$

Case 1: If $s \leq 0$, we have $u_0(s) = 0$ and $X(t,s) = s$; as a result, $s = X(t,s)$, and

$$u(x,t) = 0, \qquad \text{if } x \leq 0.$$

Case 2: If $0 < s \leq 1$, we have $u_0(s) = u_1 s$ and $X(t,x_0) = u_1 st + s$; as a result $s = X/(1 + u_1 t)$, and

$$u(x,t) = u_1 x/(1 + u_1 t), \qquad \text{if } 0 < x \leq 1 + u_1 t.$$

case 3: If $1 < s \leq 2$, we have $u_0(s) = u_1$ and $X(t,x_0) = u_1 t + s$; as a result $s = X(t,s) - u_1 t$, which implies

$$u(x,t) = u_1, \qquad \text{if } 1 + u_1 t < x \leq 2 + u_1 t.$$

Case 4: If $2 < s$, we have $u_0(s) = 0$ and $X(t,s) = u_2 t + s$; as a result $s = X(t,s)$, which implies

$$u(x,t) = 0 \qquad \text{if } 2 < x.$$

We have a shock at $x = 2$ and $t = 0$. The speed of the shock is given by the Rankin-Hugoniot formula

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \frac{\frac{1}{2}u_1^2 - \frac{1}{2}u_2^2}{u_1 - u_2} = \frac{1}{2}(u_1 + u_2).$$

As a result $x_1(t) = 2 + \frac{1}{2}(u_1 + u_2)t$. This implies that the solution is

$$u(x,t) = \begin{cases} 0, & \text{if } x \leq 0, \\ u_1 x/(1 + u_1 t), & \text{if } 0 < x \leq 1 + u_1 t, \\ u_1, & \text{if } 1 + u_1 t < x \leq 2 + \frac{1}{2}(u_1 + u_2)t, \\ u_2, & \text{if } 2 + \frac{1}{2}(u_1 + u_2)t < x. \end{cases}$$

The time T when the expansion wave catches up the shock is defined by

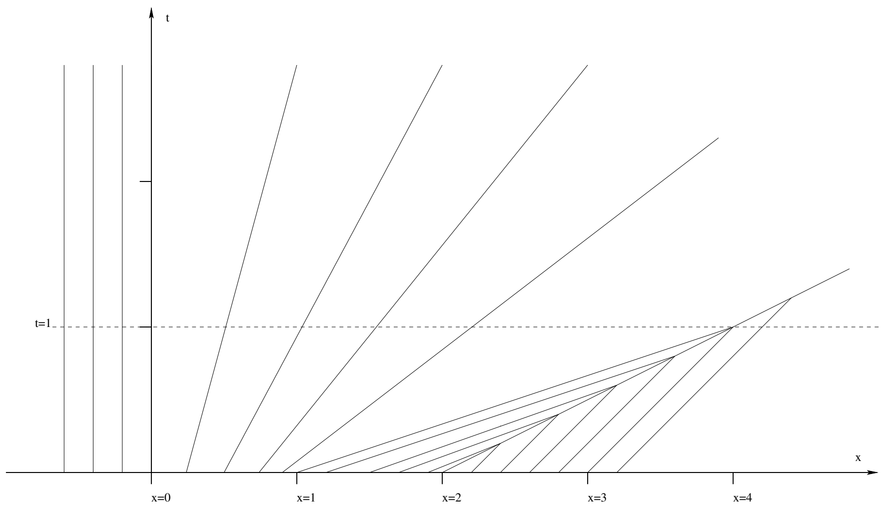
$$2 + \frac{1}{2}(u_1 + u_2)T = 1 + u_1 T,$$

that is to say

$$T = \frac{2}{u_1 - u_2}.$$

(ii) Draw the characteristics corresponding to the situation (i) with $u_1 = 2$ and $u_2 = 1$.

Solution:



(iii) Assume now that $u_1 > u_2 > u_3 = 0$. When does the first shock catches the second one?

Solution: The speed of the first shock (starting at $x = 2$ when $t = 0$) is given by the Rankin-Hugoniot formula

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \frac{\frac{1}{2}u_1^2 - \frac{1}{2}u_2^2}{u_1 - u_2} = \frac{1}{2}(u_1 + u_2).$$

As a result $x_1(t) = 2 + \frac{1}{2}(u_1 + u_2)t$. The speed of the second shock (starting at $x = 3$ when $t = 0$) is given by the Rankin-Hugoniot formula

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = \frac{\frac{1}{2}u_2^2}{u_2} = \frac{1}{2}u_2.$$

As a result $x_2(t) = 3 + \frac{1}{2}u_2 t$.

The time T' when the two shocks are at the same location is such that $x_1(T') = x_2(T')$; that is to say,

$$2 + \frac{1}{2}(u_1 + u_2)T' = 3 + \frac{1}{2}u_2 T',$$

which gives

$$T' = \frac{2}{u_1}.$$

Note that $T > T'$ for all $u_2 > 0$. This means that the first shock catches up the second one before the fans catches the first shock.

Question 98: Consider the following conservation equation

$$\partial_t u + u\partial_x u = 0, \quad x \in (-\infty, +\infty),\ t > 0, \qquad u(x,0) = u_0(x) := \begin{cases} 0 & \text{if } x \le 0, \\ x & \text{if } 0 \le x \le 1, \\ 2 - x & \text{if } 1 \le x \le 2 \\ 0 & \text{if } 2 \le x \end{cases}$$

(i) Solve this problem using the method of characteristics for $0 \le t < 1$.

Solution: The characteristics are defined by

$$\frac{dX(t,x_0)}{dt} = u(X(t,x_0),t), \qquad X(0,x_0) = x_0.$$

From class we know that $u(X(t,x_0),t)$ does not depend on time, that is to say

$$X(t,x_0) = u(X(0,x_0),0)t + x_0 = u(x_0,0)t + x_0 = u_0(x_0)t + x_0.$$

Case 1: If $x_0 \leq 0$, we have $u_0(x_0) = 0$ and $X(t, x_0) = x_0$; as a result, $x_0 = X(t, x_0)$, and

$$u(x,t) = 0, \qquad \text{if } x \leq 0.$$

Case 2: If $0 \leq x_0 \leq 1$, we have $u_0(x_0) = x_0$ and $X(t, x_0) = tx_0 + x_0$; as a result $x_0 = X/(1+t)$, and

$$u(x,t) = x/(1+t), \qquad \text{if } 0 \leq x \leq 1+t.$$

case 3: If $1 \leq x_0 \leq 2$, we have $u_0(x_0) = 2 - x_0$ and $X(t, x_0) = t(2 - x_0) + x_0$; as a result $x_0 = (X(t, x_0) - 2t)/(1-t)$, which implies

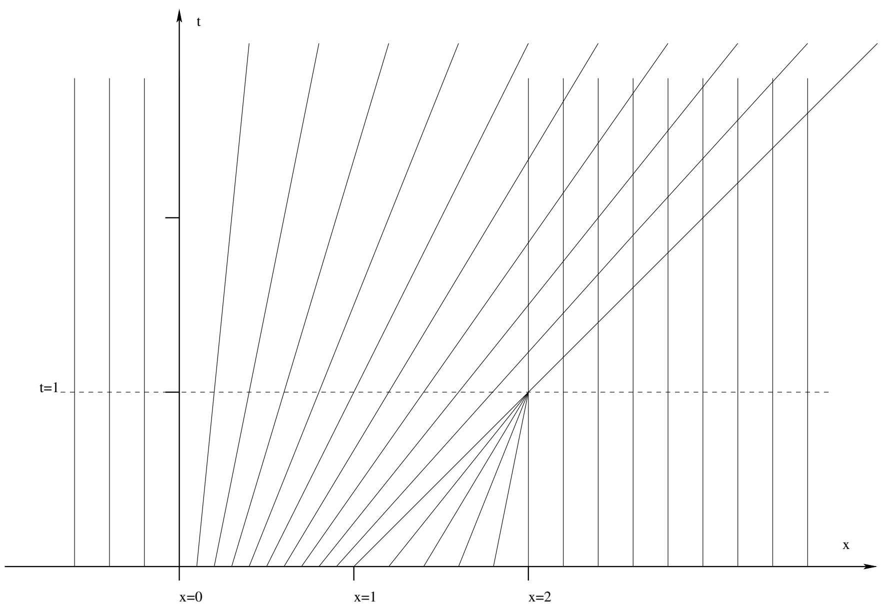
$$u(x,t) = 2 - (x - 2t)/(1-t) = (2-x)/(1-t), \qquad \text{if } 1+t \leq x \leq 2.$$

Case 4: If $2 \leq x_0$, we have $u_0(x_0) = 0$ and $X(t, x_0) = x_0$; as a result $x_0 = X(t, x_0)$, which implies

$$u(x,t) = 0 \qquad \text{if } 2 \leq x.$$

(ii) Draw the characteristics for all $t > 0$ and all $x \in \mathbb{R}$.

Solution:



(iii) There is a shock forming at $t = 1$ and $x = 2$. Let $x_s(t)$ be the location of the shock as a function of t. Compute $x_s(t)$ for $t > 1$.

Solution: Let $u^-(t)$ be the value of u at the left of the shock. Conservation of mass implies

$$\frac{1}{2} u^-(t) x_s(t) = \int_{-\infty}^{+\infty} u_0(x)dx = 1.$$

The Rankin-Hugoniot formula gives

$$\dot{x}_s(t) = \frac{\frac{1}{2}(u^-(t))^2}{u^-(t)} = \frac{1}{2} u^-(t) = \frac{1}{x_s(t)}.$$

This implies

$$x_s(t)\dot{x}_s(t) = \frac{1}{2}\frac{d}{dt}(x_s(t)^2) = 1, \quad \text{with} \quad x_s(1) = 2.$$

The Fundamental Theorem of Calculus implies

$$x_s(t)^2 - 2^2 = 2(t-1),$$

which in turn implies $x_s(t) = \sqrt{2t+2}$, for all $t \geq 1$.

(iv) Write the solution for $t > 1$.

Solution: In conclusion

$$u(x,t) = \begin{cases} 0 & \text{if } x < 0, \\ \frac{x}{1+t} & \text{if } 0 \leq x < x_s(t) = \sqrt{2t+2}, \\ 0 & \text{if } \sqrt{2t+2} = x_s(t) \leq x. \end{cases}$$

Question 99: Consider the following conservation equation

$$\partial_t u + u\partial_x u = 0, \quad x \in (-\infty, +\infty),\ t > 0, \qquad u(x,0) = u_0(x) := \begin{cases} 1 & \text{if } x \leq 0, \\ 1-x & \text{if } 0 \leq x \leq 1, \\ 0 & \text{if } 1 \leq x. \end{cases}$$

(i) Solve this problem using the method of characteristics for $0 \leq t \leq 1$.

Solution: The characteristics are defined by

$$\frac{dX(t,x_0)}{dt} = u(X(t,x_0),t), \qquad X(0,x_0) = x_0.$$

From class we know that $u(X(t,x_0),t)$ does not depend on time, that is to say

$$X(t,x_0) = u(X(0,x_0),0)t + x_0 = u(x_0,0)t + x_0 = u_0(x_0)t + x_0.$$

Case 1: If $x_0 \leq 0$, we have $u_0(x_0) = 1$ and $X(t,x_0) = t + x_0$; as a result, $x_0 = X - t$, and

$$u(x,t) = 1, \qquad \text{if } x \leq t.$$

Case 2: If $0 \leq x_0 \leq 1$, we have $u_0(x_0) = 1 - x_0$ and $X(t,x_0) = t(1-x_0) + x_0$; as a result $x_0 = (X-t)/(1-t)$, and

$$u(x,t) = 1 - (t-x)/(t-1), \qquad \text{if } 0 \leq x - t \leq 1 - t,$$

which can also be re-written

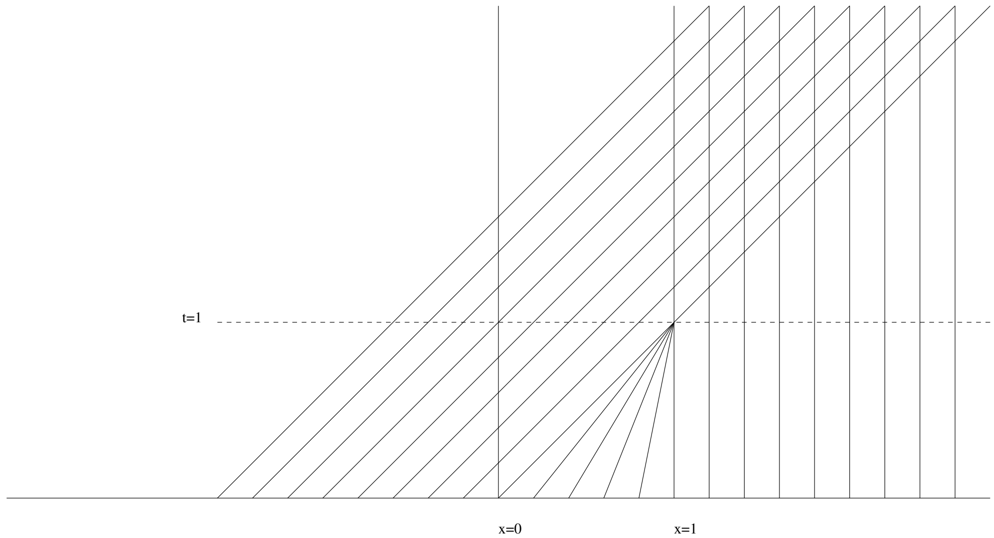
$$u(x,t) = \frac{x-1}{t-1}, \qquad \text{if } t \leq x \leq 1.$$

case 3: If $1 \leq x_0$, we have $u_0(x_0) = 0$ and $X(t,x_0) = x_0$; as a result

$$u(x,t) = 0, \qquad \text{if } 1 \leq x.$$

(ii) Draw the characteristics for all $t > 0$ and all $x \in \mathbb{R}$.

Solution:



(iii) At $t = 1$ we have $u(x, 1) = 1$ if $x < 1$ and $u(x, 1) = 0$ if $x > 1$. Solve the problem for $t > 1$.

Solution: Denote by $u_1(x)$ the solution at $t = 1$. The characteristics are $X(t, x_0) = u_1(x_0)(t - 1) + x_0$.

Case 1: If $x_0 < 1$, $u_1(x_0) = 1$ and $X(t, x_0) = t - 1 + x_0$; as a result,

$$u(x,t) = 1, \qquad \text{If } x < t.$$

Case 2: If $1 < x_0$, $u_1(x_0) = 0$ and $X(t, x_0) = x_0$; as a result,

$$u(x,t) = 0, \qquad \text{If } 1 < x.$$

The characteristics cross in the domain $\{1 < x < t\}$; as a result we have a shock. The speed of the shock is given by the Rankin-Hugoniot relation (recall that $q(u) = u^2/2$):

$$\frac{dx_s(t)}{dt} = \frac{q^+ - q^-}{u^+ - u^-} = \frac{1/2 - 0}{1 - 0} = \frac{1}{2}, \qquad x_s(1) = 1,$$

Which gives $x_s(t) = \frac{1}{2}(t + 1)$. In conclusion,

$$u(x,t) = \begin{cases} 1 & \text{If } t > 1 \text{ and } x < \frac{1}{2}(t+1), \\ 0 & \text{If } t > 1 \text{ and } \frac{1}{2}(t+1) < x. \end{cases}$$

Question 100: Give an explicit solution to the equation $\partial_t u + \partial_x(u^4) = 0$, where $x \in (-\infty, +\infty)$, $t > 0$, with initial data $u_0(x) = 0$ if $x < 0$, $u_0(x) = x^{\frac{1}{3}}$ if $0 < x < 1$, and $u_0(x) = 0$ if $1 < x$.

Solution: The implicit representation of the solution is

$$u(X(s,t),t) = u_0(s), \quad X(s,t) = s + 4u_0(s)^3 t.$$

Case 1: $s < 0$, then $u_0(s) = 0$ and $X(s,t) = s$. This means

$$u(x,t) = 0 \quad \text{if } x < 0.$$

Case 2: $0 < s < 1$, then $u_0(s) = s^{\frac{1}{3}}$ and $X(s,t) = s + 4st$. This means $s = X/(1 + 4t)$

$$u(x,t) = \left(\frac{x}{1+4t}\right)^{\frac{1}{3}} \quad \text{if } 0 < x < 1 + 4t.$$

Case 3: $1 < s$, then $u_0(s) = 0$ and $X(s,t) = s$. This means

$$u(x,t) = 0 \quad \text{if } 1 < x.$$

There is a shock starting at $x = 1$ (this is visible when one draws the characteristics).

Solution 1: The speed of the shock is given by the Rankin-Hugoniot formula

$$\frac{\mathrm{d}x_s(t)}{\mathrm{d}t} = \frac{u_+^4 - u_-^4}{u_+ - u_-}, \qquad \text{and } x_s(0) = 1,$$

where $u_+(t) = 0$ and $u_-(t) = \left(\frac{x_s(t)}{1+4t}\right)^{\frac{1}{3}}$. This gives

$$\frac{\mathrm{d}x_s(t)}{\mathrm{d}t} = u_-(t)^3 = \frac{x_s(t)}{1+4t},$$

which we re-write as follows:

$$\frac{\mathrm{d}\log(x_s(t))}{\mathrm{d}t} = \frac{1}{1+4t} = \frac{1}{4}\frac{\mathrm{d}\log(1+4t)}{\mathrm{d}t}.$$

Applying the fundamental of calculus between 0 and t gives

$$\log(x_s(t)) - \log(1) = \frac{1}{4}(\log(1+4t) - \log(1)).$$

This give

$$x_s(t) = (1+4t)^{\frac{1}{4}}.$$

Solution 2: Another (equivalent) way of solving this problem, that does not require to solve the Rankin-Hugoniot relation, consists of writing that the value of u_- is such that the total mass is conserved:

$$\int_0^{x_s(t)} u(x,t)\mathrm{d}x = \int_0^{x_s(0)} u_0(x)\mathrm{d}x = \int_0^1 x^{\frac{1}{3}}\mathrm{d}x = \frac{3}{4}$$

i.e., using the fact that $u(x,t) = (x/(1+4t))^{\frac{1}{3}}$ for all $0 \le x \le x_s(t)$, we have

$$\frac{3}{4} = (1+4t)^{-\frac{1}{3}} \int_0^{x_s(t)} x^{\frac{1}{3}}\mathrm{d}x = (1+4t)^{-\frac{1}{3}}\frac{3}{4}x_s(t)^{\frac{4}{3}}.$$

This again gives

$$x_s(t) = (1+4t)^{\frac{1}{4}}.$$

Conclusion: The solution is finally expressed as follows:

$$u(x,t) = \begin{cases} 0 & \text{if } x < 0 \\ \left(\frac{x}{1+4t}\right)^{\frac{1}{3}} & \text{if } 0 < x < (1+4t)^{\frac{1}{4}} \\ 0 & \text{if } (1+4t)^{\frac{1}{4}} < x \end{cases}$$