4 Fourier transform

Here are some formulae that you may want to use:

$$\mathcal{F}(f)(\omega) \stackrel{\text{def}}{=} \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x)e^{i\omega x}dx, \qquad \mathcal{F}^{-1}(f)(x) = \int_{-\infty}^{+\infty} f(\omega)e^{-i\omega x}d\omega, \tag{3}$$

$$\mathcal{F}(f*g) = 2\pi\mathcal{F}(f)\mathcal{F}(g), \tag{4}$$

$$\mathcal{F}(e^{-\alpha|x|}) = \frac{1}{\pi}\frac{\alpha}{\omega^2+\alpha^2}, \qquad \mathcal{F}(\frac{2\alpha}{x^2+\alpha^2})(\omega) = e^{-\alpha|\omega|}, \tag{5}$$

$$\mathcal{F}(f(x-\beta))(\omega) = e^{i\beta\omega}\mathcal{F}(f)(\omega), \tag{6}$$

$$\mathcal{F}(e^{-\alpha x^2}) = \frac{1}{\sqrt{4\pi\alpha}}e^{-\frac{\omega^2}{4\alpha}} \tag{7}$$

$$\mathcal{F}(H(x)e^{-ax})(\omega) = \frac{1}{2\pi}\frac{1}{a-i\omega}, \qquad H \text{ is the Heaviside function} \tag{8}$$

$$\mathcal{F}(\mathrm{pv}(\frac{1}{x}))(\omega) = \frac{i}{2}\mathrm{sign}(\omega) = \frac{i}{2}(H(\omega)-H(-\omega)) \tag{9}$$

$$\mathcal{F}(\mathrm{sech}(ax)) = \frac{1}{2a}\mathrm{sech}(\frac{\pi}{2a}\omega), \qquad \mathrm{sech}(ax) \stackrel{\text{def}}{=} \frac{1}{\mathrm{ch}(x)} = \frac{2}{e^x+e^{-x}} \tag{10}$$

$$\cos(a)-\cos(b) = -2\sin(\tfrac{1}{2}(a+b))\sin(\tfrac{1}{2}(a-b)) \tag{11}$$

Question 41: (i) Let f be an integrable function on $(-\infty,+\infty)$. Prove that for all $a,b\in\mathbb{R}$, and for all $\xi\in\mathbb{R}$, $\mathcal{F}([e^{ibx}f(ax)])(\xi) = \frac{1}{a}\mathcal{F}(f)(\frac{\xi+b}{a})$.

Solution: The definition of the Fourier transform together with the change of variable $ax \longmapsto x'$ implies

$$\begin{aligned}\mathcal{F}[e^{ibx}f(ax)])(\xi) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(ax)e^{ibx}e^{i\xi x}dx\\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(ax)e^{i(b+\xi)x}dx\\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{1}{a}f(x')e^{i\frac{(\xi+b)}{a}x'}dx'\\ &= \tfrac{1}{a}\mathcal{F}(f)(\tfrac{\xi+b}{a}).\end{aligned}$$

(ii) Let c be a positive real number. Compute the Fourier transform of $f(x) = e^{-cx^2}\sin(bx)$.

Solution: Using the fact that $\sin(bx) = -i\frac{1}{2}(e^{ibx}-e^{-ibx})$, and setting $a=\sqrt{c}$ we infer that

$$\begin{aligned}f(x) &= \tfrac{1}{2i}e^{-(ax)^2}(e^{ibx}-e^{-ibx})\\ &= \tfrac{1}{2i}e^{-(ax)^2}e^{ibx} - \tfrac{1}{2i}e^{-(ax)^2}e^{i(-b)x})\end{aligned}$$

and using (i) and (7) we deduce

$$\hat{f}(\xi) = \tfrac{1}{2i}\tfrac{1}{a}\tfrac{1}{\sqrt{4\pi}}(e^{-(\frac{\xi+b}{a})^2} - e^{-(\frac{\xi-b}{a})^2}).$$

In conclusion

$$\hat{f}(\xi) = \tfrac{1}{2i}\tfrac{1}{\sqrt{4\pi c}}(e^{-\frac{1}{c}(\xi+b)^2} - e^{-\frac{1}{c}(\xi-b)^2}).$$

Question 42: (a) Compute the Fourier transform of the function $f(x)$ defined by

$$f(x) = \begin{cases}1 & \text{if } |x|\le 1\\ 0 & \text{otherwise}\end{cases}$$

Solution: By definition

$$\mathcal{F}(f)(\omega) = \frac{1}{2\pi}\int_{-1}^{1} e^{i\xi\omega} = = \frac{1}{2\pi}\frac{1}{i\omega}(e^{i\omega} - e^{-i\omega})$$
$$= \frac{1}{2\pi}\frac{2\sin(\omega)}{\omega}.$$

Hence

$$\mathcal{F}(f)(\omega) = \frac{1}{\pi}\frac{\sin(\omega)}{\omega}.$$

(b) Find the inverse Fourier transform of $g(\omega) = \frac{\sin(\omega)}{\omega}$.

Solution: Using (a) we deduce that $g(\omega) = \pi\mathcal{F}(f)(\omega)$, that is to say, $\mathcal{F}^{-1}(g)(x) = \pi\mathcal{F}^{-1}(\mathcal{F}(f))(x)$. Now, using the inverse Fourier transform, we deduce that $\mathcal{F}^{-1}(g)(x) = \pi f(x)$ at every point x where $f(x)$ is of class C^1 and $\mathcal{F}^{-1}(g)(x) = \frac{\pi}{2}(f(x^-) + f(x^+))$ at discontinuity points of f. As a result:

$$\mathcal{F}^{-1}(g)(x) = \begin{cases} \pi & \text{if } |x| < 1 \\ \frac{\pi}{2} & \text{at } |x| = 1 \\ 0 & \text{otherwise} \end{cases}$$

Question 43: Use the Fourier transform technique to solve the following PDE:

$$\partial_t u(x,t) + c\partial_x u(x,t) + \gamma u(x,t) = 0,$$

for all $x \in (-\infty, +\infty)$, $t > 0$, with $u(x,0) = u_0(x)$ for all $x \in (-\infty, +\infty)$.

Solution: By taking the Fourier transform of the PDE, one obtains

$$\partial_t \mathcal{F}(u) - i\omega c\mathcal{F}(y) + \gamma\mathcal{F}(y) = 0.$$

The solution is

$$\mathcal{F}(u)(\omega,t) = c(\omega)e^{i\omega ct - \gamma t}.$$

The initial condition implies that $c(\omega) = \mathcal{F}(u_0)(\omega)$:

$$\mathcal{F}(u)(\omega,t) = \mathcal{F}(u_0)(\omega)e^{i\omega ct}e^{-\gamma t}.$$

The shift lemma in turn implies that

$$\mathcal{F}(u)(\omega,t) = \mathcal{F}(u_0(x-ct))(\omega)e^{-\gamma t} = \mathcal{F}(u_0(x-ct)e^{-\gamma t})(\omega).$$

Applying the inverse Fourier transform gives:

$$u(x,t) = u_0(x-ct)e^{-\gamma t}.$$

Question 44: Solve by the Fourier transform technique the following equation: $\partial_{xx}\phi(x) - 2\partial_x\phi(x) + \phi(x) = H(x)e^{-x}$, $x \in (-\infty, +\infty)$, where $H(x)$ is the Heaviside function. (Hint: use the factorization $i\omega^3 + \omega^2 + i\omega + 1 = (1+\omega^2)(1+i\omega)$ and recall that $\mathcal{F}(f(x))(-\omega) = \mathcal{F}(f(-x))(\omega)$).

Solution: Applying the Fourier transform with respect to x gives

$$(-\omega^2 + 2i\omega + 1)\mathcal{F}(\phi)(\omega) = \mathcal{F}(H(x)e^{-x})(\omega) = \frac{1}{2\pi}\frac{1}{1-i\omega}.$$

where we used (8). Then, using the hint gives

$$\mathcal{F}(\phi)(\omega) = \frac{1}{2\pi}\frac{1}{(1-i\omega)(-\omega^2+2i\omega+1)} = \frac{1}{2\pi}\frac{1}{i\omega^3+\omega^2+i\omega+1}$$
$$= \frac{1}{2\pi}\frac{1}{1+\omega^2}\frac{1}{1+i\omega}.$$

We now use again (8) and (5) to obtain

$$\mathcal{F}(\phi)(\omega) = \pi\frac{1}{\pi}\frac{1}{1+\omega^2}\frac{1}{2\pi}\frac{1}{1-i(-\omega)} = \pi\mathcal{F}(e^{-|x|})(\omega)\mathcal{F}(H(x)e^{-x})(-\omega).$$

Now we use $\mathcal{F}(H(x)e^{-x})(-\omega) = \mathcal{F}(H(-x)e^{x})(\omega)$ and we finally have

$$\mathcal{F}(\phi)(\omega) = \pi\mathcal{F}(e^{-|x|})(\omega)\mathcal{F}(H(-x)e^{x})(\omega).$$

The Convolution Theorem (4) gives

$$\mathcal{F}(\phi)(\omega) = \pi\frac{1}{2\pi}\mathcal{F}(e^{-|x|} * (H(-x)e^{x}))(\omega).$$

We obtain ϕ by using the inverse Fourier transform

$$\phi(x) = \frac{1}{2}e^{-|x|} * (H(-x)e^{x}),$$

i.e.,

$$\phi(x) = \int_{-\infty}^{+\infty} \frac{1}{2}e^{-|x-y|}H(-y)e^{y}\text{d}y$$

and recalling that H is the Heaviside function we finally have

$$\phi(x) = \frac{1}{2}\int_{-\infty}^{0} e^{y-|x-y|}\text{d}y = \begin{cases} \frac{1}{4}e^{-x} & \text{if } x \geq 0 \\ (\frac{1}{4} - x)e^{x} & \text{if } x \leq 0. \end{cases}$$

Question 45: Use the Fourier transform technique to solve the following ODE $y''(x) - y(x) = f(x)$ for $x \in (-\infty, +\infty)$, with $y(\pm\infty) = 0$, where f is a function such that $|f|$ is integrable over $\mathbb{R}$.

Solution: By taking the Fourier transform of the ODE, one obtains

$$-\omega^2\mathcal{F}(y) - \mathcal{F}(y) = \mathcal{F}(f).$$

That is

$$\mathcal{F}(y) = -\mathcal{F}(f)\frac{1}{1+\omega^2}.$$

and the convolution Theorem, see (4), together with (5) gives

$$\mathcal{F}(y) = -\pi\mathcal{F}(f)\mathcal{F}(e^{-|x|}) = -\frac{1}{2}\mathcal{F}(f * e^{-|x|}).$$

Applying $\mathcal{F}^{-1}$ on both sides we obtain

$$y(x) = -\frac{1}{2}f * e^{-|x|} = -\frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-z|}f(z)dz$$

That is

$$y(x) = -\frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-z|}f(z)dz.$$

Question 46: Use the Fourier transform method to compute the solution of $u_{tt} - a^2u_{xx} = 0$, where $x \in \mathbb{R}$ and $t \in (0, +\infty)$, with $u(0, x) = f(x) := \sin^2(x)$ and $u_t(0, x) = 0$ for all $x \in \mathbb{R}$.

Solution: Take the Fourier transform in the x direction:

$$\mathcal{F}(u)_{tt} + \omega^2a^2\mathcal{F}(u) = 0.$$

This is an ODE. The solution is

$$\mathcal{F}(u)(t, \omega) = c_1(\omega)\cos(\omega at) + c_2(\omega)\sin(\omega at).$$

The initial boundary conditions give

$$\mathcal{F}(u)(0,\xi) = \mathcal{F}(f)(\omega) = c_1(\omega)$$

and $c_2(\omega) = 0$. Hence

$$\mathcal{F}(t,\omega) = \mathcal{F}(f)(\omega)\cos(\omega a t) = \frac{1}{2}\mathcal{F}(f)(\omega)(e^{ia\omega t} + e^{-ia\omega t}).$$

Using the shift lemma, we infer that

$$\mathcal{F}(t,\omega) = \frac{1}{2}(\mathcal{F}(f(x-at))(\omega) + \mathcal{F}(f(x+at))(\omega)).$$

Usin the inverse Fourier transform, we finaly conclude that

$$u(t,x) = \frac{1}{2}(f(x-at) + f(x+at)) = \frac{1}{2}(\sin^2(x+at) + \sin^2(x-at)).$$

Note that this is the D'Alembert formula.

Question 47: Use the Fourier transform method to compute the solution of $u_{tt} - a^2 u_{xx} = 0$, where $x \in \mathbb{R}$ and $t \in (0,+\infty)$, with $u(0,x) = f(x) := \cos^2(x)$ and $u_t(0,x) = 0$ for all $x \in \mathbb{R}$.

Solution: Take the Fourier transform in the x direction:

$$\mathcal{F}(u)_{tt} + \omega^2 a^2 \mathcal{F}(u) = 0.$$

This is an ODE. The solution is

$$\mathcal{F}(u)(t,\omega) = c_1(\omega)\cos(\omega a t) + c_2(\omega)\sin(\omega a t).$$

The initial boundary conditions give

$$\mathcal{F}(u)(0,\xi) = \mathcal{F}(f)(\omega) = c_1(\omega)$$

and $c_2(\omega) = 0$. Hence

$$\mathcal{F}(t,\omega) = \mathcal{F}(f)(\omega)\cos(\omega a t) = \frac{1}{2}\mathcal{F}(f)(\omega)(e^{ia\omega t} + e^{-ia\omega t}).$$

Using the shift lemma (i.e., formula (6)) we obtain

$$u(t,x) = \frac{1}{2}(f(x-at) + f(x+at)) = \frac{1}{2}(\cos^2(x+at) + \cos^2(x-at)).$$

Note that this is the D'Alembert formula.

Question 48: Solve the integral equation: $f(x) + \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{f(y)}{(x-y)^2+1} dy = \frac{1}{x^2+4} + \frac{1}{x^2+1}$, for all $x \in (-\infty,+\infty)$.

Solution: The equation can be re-written

$$f(x) + \frac{1}{2\pi} f * \frac{1}{x^2+1} = \frac{1}{x^2+4} + \frac{1}{x^2+1}.$$

We take the Fourier transform of the equation and we apply the Convolution Theorem (see (4))

$$\mathcal{F}(f) + \frac{1}{2\pi} 2\pi \mathcal{F}(\frac{1}{x^2+1})\mathcal{F}(f) = \mathcal{F}(\frac{1}{x^2+4}) + \mathcal{F}(\frac{1}{x^2+1}).$$

Using (5), we obtain

$$\mathcal{F}(f) + \frac{1}{2}e^{-|\omega|}\mathcal{F}(f) = \frac{1}{4}e^{-2|\omega|} + \frac{1}{2}e^{-|\omega|},$$

which gives

$$\mathcal{F}(f)(1 + \frac{1}{2}e^{-|\omega|}) = \frac{1}{2}e^{-|\omega|}(\frac{1}{2}e^{-|\omega|} + 1).$$

We then deduce

$$\mathcal{F}(f) = \frac{1}{2}e^{-|\omega|}.$$

Taking the inverse Fourier transform, we finally obtain $f(x) = \frac{1}{x^2+1}$.

Question 49: Solve the following integral equation (Hint: solution is short):

$$\int_{-\infty}^{+\infty} f(y)f(x-y)\mathrm{d}y - 2\sqrt{2}\int_{-\infty}^{+\infty} e^{-\frac{y^2}{2\pi}} f(x-y)\mathrm{d}y = -2\pi e^{-\frac{x^2}{4\pi}}. \qquad \forall x \in \mathbb{R}.$$

Solution: This equation can be re-written using the convolution operator:

$$f * f - 2\sqrt{2}e^{-\frac{x^2}{2\pi}} * f = -2\pi e^{-\frac{x^2}{4\pi}}.$$

We take the Fourier transform and use the convolution theorem (4) together with (7) to obtain

$$\begin{aligned} 2\pi\mathcal{F}(f)^2 - 2\pi 2\sqrt{2}\mathcal{F}(f)\frac{1}{\sqrt{4\pi\frac{1}{2\pi}}}e^{-\omega^2\frac{1}{4\frac{1}{2\pi}}} &= -2\pi\frac{1}{\sqrt{4\pi\frac{1}{4\pi}}}e^{-\omega^2\frac{1}{4\frac{1}{4\pi}}} \\ \mathcal{F}(f)^2 - 2\mathcal{F}(f)e^{-\omega^2\frac{\pi}{2}} + e^{-\omega^2\pi} &= 0 \\ (\mathcal{F}(f) - e^{-\omega^2\frac{\pi}{2}})^2 &= 0. \end{aligned}$$

This implies

$$\mathcal{F}(f) = e^{-\omega^2\frac{\pi}{2}}.$$

Taking the inverse Fourier transform, we obtain

$$f(x) = \sqrt{2}e^{-\frac{x^2}{2\pi}}.$$

Question 50: Solve the following integral equation (Hint: $x^2 - 3xa + 2a^2 = (x-a)(x-2a)$):

$$\int_{-\infty}^{+\infty} f(y)f(x-y)\mathrm{d}y - 3\sqrt{2}\int_{-\infty}^{+\infty} e^{-\frac{y^2}{2\pi}} f(x-y)\mathrm{d}y = -4\pi e^{-\frac{x^2}{4\pi}}. \qquad \forall x \in \mathbb{R}.$$

Solution: This equation can be re-written using the convolution operator:

$$f * f - 3\sqrt{2}e^{-\frac{x^2}{2\pi}} * f = -4\pi e^{-\frac{x^2}{4\pi}}.$$

We take the Fourier transform and use (7) to obtain

$$\begin{aligned} 2\pi\mathcal{F}(f)^2 - 2\pi 3\sqrt{2}\mathcal{F}(f)\frac{1}{\sqrt{4\pi\frac{1}{2\pi}}}e^{-\omega^2\frac{1}{4\frac{1}{2\pi}}} &= -4\pi\frac{1}{\sqrt{4\pi\frac{1}{4\pi}}}e^{-\omega^2\frac{1}{4\frac{1}{4\pi}}} \\ \mathcal{F}(f)^2 - 3\mathcal{F}(f)e^{-\omega^2\frac{\pi}{2}} + 2e^{-\omega^2\pi} &= 0 \\ (\mathcal{F}(f) - e^{-\omega^2\frac{\pi}{2}})(\mathcal{F}(f) - 2e^{-\omega^2\frac{\pi}{2}}) &= 0. \end{aligned}$$

This implies

$$\text{either} \quad \mathcal{F}(f) = e^{-\omega^2\frac{\pi}{2}}, \quad \text{or} \quad \mathcal{F}(f) = 2e^{-\omega^2\frac{\pi}{2}}.$$

Taking the inverse Fourier transform, we obtain

$$\text{either} \quad f(x) = \sqrt{2}e^{-\frac{x^2}{2\pi}}, \quad \text{or} \quad f(x) = 2\sqrt{2}e^{-\frac{x^2}{2\pi}}.$$

Question 51: Solve the integral equation: $f(x) + \frac{3}{2}\int_{-\infty}^{+\infty} e^{-|x-y|}f(y)dy = e^{-|x|}$, for all $x \in (-\infty, +\infty)$.

Solution: The equation can be re-written

$$f(x) + \frac{3}{2}e^{-|x|} * f = e^{-|x|}.$$

We take the Fourier transform of the equation and apply the Convolution Theorem (see (4))

$$\mathcal{F}(f) + \frac{3}{2}2\pi\mathcal{F}(e^{-|x|})\mathcal{F}(f) = \mathcal{F}(e^{-|x|}).$$

Using (5), we obtain

$$\mathcal{F}(f) + 3\pi\frac{1}{\pi}\frac{1}{1+\omega^2}\mathcal{F}(f) = \frac{1}{\pi}\frac{1}{1+\omega^2},$$

which gives

$$\mathcal{F}(f)\frac{\omega^2+4}{1+\omega^2} = \frac{1}{\pi}\frac{1}{1+\omega^2}.$$

We then deduce

$$\mathcal{F}(f) = \frac{1}{\pi}\frac{1}{4+\omega^2} = \frac{1}{2}\mathcal{F}(e^{-2|x|}).$$

Taking the inverse Fourier transform, we finally obtain $f(x) = \frac{1}{2}e^{-2|x|}$.

Question 52: Solve the following integral equation $\int_{-\infty}^{+\infty} e^{-(x-y)^2}g(y)dy = e^{-\frac{1}{2}x^2}$ for all $x \in (-\infty, +\infty)$, i.e., find the function g that solves the above equation.

Solution: The left-hand side of the equation is a convolution; hence,

$$e^{-x^2} * g(x) = e^{-\frac{1}{2}x^2}.$$

By taking the Fourier transform, we obtain

$$2\pi\frac{1}{\sqrt{4\pi}}e^{-\frac{1}{4}\omega^2}\mathcal{F}g(\omega) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\omega^2}.$$

That gives

$$\mathcal{F}(g)(\omega) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{4}\omega^2}.$$

By taking the inverse Fourier transform, we deduce

$$g(x) = \frac{\sqrt{4\pi}}{\sqrt{2\pi}}e^{-x^2} = \sqrt{\frac{2}{\pi}}e^{-x^2}.$$

Question 53: Solve the integral equation: $\int_{-\infty}^{+\infty} f(y)f(x-y)dy = \frac{4}{x^2+4}$, for all $x \in (-\infty, +\infty)$. How many solutions did you find?

Solution: The equation can be re-written

$$f * f = \frac{4}{x^2+4}.$$

We take the Fourier transform of the equation and apply the Convolution Theorem (see (4))

$$2\pi\mathcal{F}(f)^2 = \mathcal{F}(\frac{4}{x^2+4})$$

Using (5), we obtain

$$2\pi\mathcal{F}(f)^2 = e^{-2|\omega|}.$$

which gives

$$\mathcal{F}(f) = \pm\frac{1}{\sqrt{2\pi}}e^{-|\omega|}.$$

Taking the inverse Fourier transform, we finally obtain

$$f(x) = \pm\frac{1}{\sqrt{2\pi}}\frac{2}{x^2+1}.$$

We found two solutions: a positive one and a negative one.

Question 54: Use the Fourier transform method to solve the equation $\partial_t u + \frac{2t}{1+t^2}\partial_x u = 0$, $u(x,0) = u_0(x)$, in the domain $x \in (-\infty, +\infty)$ and $t > 0$.

Solution: We take the Fourier transform of the equation with respect to x

$$\begin{aligned} 0 &= \partial_t \mathcal{F}(u) + \mathcal{F}(\frac{2t}{1+t^2}\partial_x u) \\ &= \partial_t \mathcal{F}(u) + \frac{2t}{1+t^2}\mathcal{F}(\partial_x u) \\ &= \partial_t \mathcal{F}(u) - i\omega\frac{2t}{1+t^2}\mathcal{F}(u). \end{aligned}$$

This is a first-order linear ODE:

$$\frac{\partial_t \mathcal{F}(u)}{\mathcal{F}(u)} = i\omega\frac{2t}{1+t^2} = i\omega\frac{\mathsf{d}}{\mathsf{d}t}(\log(1+t^2))$$

The solution is

$$\mathcal{F}(u)(\omega,t) = K(\omega)e^{i\omega\log(1+t^2)}.$$

Using the initial condition, we obtain

$$\mathcal{F}(u_0)(\omega) = \mathcal{F}(u)(\omega,0) = K(\omega).$$

The shift lemma (see formula (6)) implies

$$\mathcal{F}(u)(\omega,t) = \mathcal{F}(u_0)(\omega)e^{i\omega\log(1+t^2)} = \mathcal{F}(u_0(x - \log(1+t^2))),$$

Applying the inverse Fourier transform finally gives

$$u(x,t) = u_0(x - \log(1+t^2)).$$

Question 55: Solve the integral equation:

$$\int_{-\infty}^{+\infty}\left(f(y) - \sqrt{2}e^{-\frac{y^2}{2\pi}} - \frac{1}{1+y^2}\right)f(x-y)dy = -\int_{-\infty}^{+\infty}\frac{\sqrt{2}}{1+y^2}e^{-\frac{(x-y)^2}{2\pi}}\mathrm{d}y, \qquad \forall x \in (-\infty,+\infty).$$

(Hint: there is an easy factorization after applying the Fourier transform.)

Solution: The equation can be re-written

$$f * (f - \sqrt{2}e^{-\frac{x^2}{2\pi}} - \frac{1}{1+x^2}) = -\frac{1}{1+x^2} * \sqrt{2}e^{-\frac{x^2}{2\pi}}.$$

We take the Fourier transform of the equation and apply the Convolution Theorem (see (4))

$$2\pi\mathcal{F}(f)\left(\mathcal{F}(f) - \sqrt{2}\mathcal{F}(e^{-\frac{x^2}{2\pi}}) - \mathcal{F}(\frac{1}{1+x^2})\right) = -2\pi\mathcal{F}(\frac{1}{1+x^2})\sqrt{2}\mathcal{F}(e^{-\frac{x^2}{2\pi}})$$

Using (5), (7) we obtain

$$\begin{aligned} \sqrt{2}\mathcal{F}(e^{-\frac{x^2}{2\pi}}) &= \sqrt{2}\frac{1}{\sqrt{4\pi\frac{1}{2\pi}}}e^{-\frac{\omega^2}{4\frac{1}{2\pi}}} = e^{-\frac{\pi\omega^2}{2}} \\ \mathcal{F}(\frac{1}{1+x^2}) &= \frac{1}{2}e^{-|\omega|}, \end{aligned}$$

which gives

$$\mathcal{F}(f)\left(\mathcal{F}(f) - e^{-\frac{\pi\omega^2}{2}} - \frac{1}{2}e^{-|\omega|}\right) = -\frac{1}{2}e^{-|\omega|}e^{-\frac{\pi\omega^2}{2}}.$$

This equation can also be re-written as follows

$$\mathcal{F}(f)^2 - \mathcal{F}(f)e^{-\frac{\pi\omega^2}{2}} - \mathcal{F}(f)\frac{1}{2}e^{-|\omega|} + \frac{1}{2}e^{-|\omega|}e^{-\frac{\pi\omega^2}{2}} = 0,$$

and can be factorized as follows:

$$(\mathcal{F}(f) - e^{-\frac{\pi\omega^2}{2}})(\mathcal{F}(f) - \frac{1}{2}e^{-|\omega|}) = 0.$$

This means that either $\mathcal{F}(f) = e^{-\frac{\pi\omega^2}{2}}$ or $\mathcal{F}(f) = \frac{1}{2}e^{-|\omega|}$. Taking the inverse Fourier transform, we finally obtain two solutions

$$f(x) = \sqrt{2}e^{-\frac{x^2}{2\pi}}, \quad \text{or} \quad f(x) = \frac{1}{1+x^2}.$$

Another solution consists of observing that the equation can also be re-written

$$\mathcal{F}(f)^2 - \sqrt{2}\mathcal{F}(e^{-\frac{x^2}{2\pi}})\mathcal{F}(f) - \mathcal{F}(\frac{1}{1+x^2})\mathcal{F}(f) + \mathcal{F}(\frac{1}{1+x^2})\sqrt{2}\mathcal{F}(e^{-\frac{x^2}{2\pi}}) = 0$$