8 Greens function

Question 101: Let Ω be a three-dimensional domain and consider the PDE

$$\nabla^2 u = f(x), \quad x \in \Omega, \quad \text{with} \quad u(x) = h(x) \quad \text{on the boundary of } \Omega, \text{ say } \Gamma.$$

Let $G(x, x_0)$ be the Green's function of this problem (the exact expression of G does not matter; just assume that G is known). Give a representation[1] of $u(x)$ in terms of G, f and h.

Solution: By definition

$$\nabla_x^2 G(x, x_0) = \delta(x - x_0), \quad x \in \Omega, \quad \text{with} \quad G(x, x_0) = 0 \quad x \in \Gamma.$$

Then using the integration by parts formula, we obtain

$$\int_\Omega u(x)\nabla_x^2(G(x,x_0))dx = \int_\Omega \nabla_x^2(u(x))G(x,x_0)dx + \int_\Gamma u(x)\partial_n(G(x,x_0))dx - \int_\Gamma \partial_n(u(x))G(x,x_0)dx.$$

which can also be rewritten

$$u(x_0) = \int_\Omega f(x)G(x,x_0)dx + \int_\Gamma h(x)\partial_n(G(x,x_0))dx.$$

Question 102: Let f be a smooth function in $[0,1]$. Consider the PDE

$$u - \partial_{xx}u = f(x), \quad x \in (0,1), \qquad \partial_x u(1) + u(1) = 2, \ -\partial_x u(0) + u(0) = 1.$$

What PDE and which boundary conditions must satisfy the Green function, $G(x, x_0)$, (DO NOT compute the Green function)? Give the integral representation of u assuming $G(x, x_0)$ is known. Fully justify your answer.

Solution: Multiply the equation by $G(x, x_0)$ and integrate over $(0,1)$:

$$\begin{aligned}
\int_0^1 f(x)G(x,x_0)dx &= \int_0^1 (u(x) - \partial_{xx}u(x))G(x,x_0)dx \\
&= \int_0^1 u(x)G(x,x_0) + \partial_x u(x)\partial_x G(x,x_0)dx - \partial_x u(1)G(1,x_0) + \partial_x u(0)G(0,x_0) \\
&= \int_0^1 u(x)(G(x,x_0) - \partial_{xx}G(x,x_0))dx + u(1)\partial_x G(1,x_0) - u(0)\partial_x G(0,x_0) \\
&\quad - \partial_x u(1)G(1,x_0) + \partial_x u(0)G(0,x_0) \\
&= \int_0^1 u(x)(G(x,x_0) - \partial_{xx}G(x,x_0))dx + u(1)\partial_x G(1,x_0) - u(0)\partial_x G(0,x_0) \\
&\quad (u(1)-2)G(1,x_0) + (u(0)-1)G(0,x_0) \\
&= \int_0^1 u(x)(G(x,x_0) - \partial_{xx}G(x,x_0))dx \\
&\quad + u(1)(G(1,x_0) + \partial_x G(1,x_0)) + u(0)(G(0,x_0) - \partial_x G(0,x_0)) - 2G(1,x_0) - G(0,x_0)
\end{aligned}$$

If we define $G(x, x_0)$ so that

$$G(x,x_0) - \partial_{xx}G(x,x_0) = \delta(x - x_0), \qquad G(1,x_0) + \partial_x G(1,x_0) = 0, \ G(0,x_0) - \partial_x G(0,x_0) = 0,$$

then $u(x_0)$, $x_0 \in (0,1)$, has the following representation

$$u(x_0) = \int_0^1 f(x)G(x,x_0)dx + 2G(1,x_0) + G(0,x_0), \qquad \forall x_0 \in (0,1).$$

[1] Hint: use $\int_\Omega \psi\nabla^2(\phi) = \int_\Omega \nabla^2(\psi)\phi + \int_\Gamma \psi\partial_n(\phi) - \int_\Gamma \partial_n(\psi)\phi$

Question 103: Consider the equation $u'(x) + u = f(x)$ for $x \in (0,1)$ with $u(0) = a$. Let $G(x, x_0)$ be the associated Green's function. (Pay attention to the number of derivatives).
(a) Give the equation and boundary condition defining G and give an integral representation of $u(x_0)$ in terms of G, f and the boundary data a. (Do not compute G.)

Solution: The Green's function is defined by

$$-G'(x, x_0) + G(x, x_0) = \delta(x - x_0), \quad G(1, x_0) = 0.$$

We multiply the equation by u and we integrate over $(0,1)$ (in the distribution sense),

$$\int_0^1 -G'(x, x_0)u(x)\mathsf{d}x + \int_0^1 G(x, x_0)u(x)\mathsf{d}x = u(x_0).$$

We integrates by parts and we obtain,

$$u(x_0) = \int_0^1 G(x, x_0)(u'(x) + u(x))\mathsf{d}x - G(1, x_0)u(1) + G(0, x_0)u(0)$$

Then, using the fact that $u' + u = f$ and using the boundary conditions for G and u, we obtain

$$u(x_0) = \int_0^1 G(x, x_0)f(x)\mathsf{d}x + 2G(0, x_0). \quad \forall x_0 \in (0,1).$$

(b) Compute $G(x,x_0)$.

Solution: For $x<x_0$ and $x_0>x$ we have

$$-G'(x,x_0)+G(x,x_0)=0.$$

The solution is

$$G(x,x_0)=\begin{cases}\alpha e^x & \text{for } x<x_0\\ \beta e^x & \text{for } x>x_0.\end{cases}$$

The boundary condition $G(1,x_0)=0$ implies $\beta=0$.

For every $\epsilon>0$ we have

$$\begin{aligned}1&=\int_{x_0-\epsilon}^{x_0+\epsilon}(-G'(x,x_0)+G(x,x_0))\mathsf{d}x\\ &=G(x_0-\epsilon,x_0)-G(x_0+\epsilon,x_0)+\int_{x_0-\epsilon}^{x_0+\epsilon}G(x,x_0)\mathsf{d}x\end{aligned}$$

The term $R_\epsilon=\int_{x_0-\epsilon}^{x_0+\epsilon}G(x,x_0)\mathsf{d}x$ can be bounded as follows:

$$|R_\epsilon|\le 2\epsilon\max_{x\in[0,1]}|G(x,x_0)|=2\epsilon\alpha e^{x_0}.$$

Clearly R_ϵ goes to 0 with ϵ. As a result we obtain the jump condition

$$1=G(x_0^-,x_0)-G(x_0^+,x_0)=\alpha e^{x_0}.$$

This implies

$$\alpha=e^{-x_0}.$$

Finally

$$G(x,x_0)=\begin{cases}e^{x-x_0} & \text{for } x<x_0\\ 0 & \text{for } x>x_0.\end{cases}$$

Question 104: Consider the equation $-\partial_x(x\partial_x u(x))=f(x)$ for all $x\in(1,2)$ with $u(1)=a$ and $u(2)=b$. Let $G(x,x_0)$ be the associated Green's function.
(i) Give the equation and boundary conditions satisfied by G and give the integral representation of $u(x_0)$ for all $x_0\in(1,2)$ in terms of G, f, and the boundary data. (Do not compute G in this question).

Solution: We have a second-order PDE and the operator is clearly self-adjoint. The Green's function solves the equation

$$-\partial_x(x\partial_x G(x,x_0))=\delta(x-x_0),\quad G(1,x_0)=0,\quad G(2,x_0)=0.$$

We multiply the equation by u and integrate over the domain $(1,2)$ (in the distribution sense).

$$\langle\delta(x-x_0),u\rangle=u(x_0)=-\int_1^2\partial_x(x\partial_x G(x,x_0))u(x)\mathsf{d}x.$$

We integrate by parts and we obtain,

$$\begin{aligned}u(x_0)&=\int_1^2 x\partial_x G(x,x_0)\partial_x u(x)\mathsf{d}x-[x\partial_x G(x,x_0)u(x)]_1^2\\ &=-\int_1^2 G(x,x_0)\partial_x(x\partial_x u(x))\mathsf{d}x-2\partial_x G(2,x_0)u(2)+\partial_x G(1,x_0)u(1).\end{aligned}$$

Now, using the boundary conditions and the fact that $-\partial_x(x\partial_x u(x))=f(x)$, we finally have

$$u(x_0)=\int_1^2 G(x,x_0)f(x)\mathsf{d}x-2\partial_x G(2,x_0)b+\partial_x G(1,x_0)a.$$

(ii) Compute $G(x,x_0)$ for all $x,x_0 \in (1,2)$.

Solution: For all $x \neq x_0$ we have

$$-\partial_x(x\partial_x G(x,x_0)) = 0.$$

The solution is

$$G(x,x_0) = \begin{cases} a\log(x) + b & \text{if } 1 < x < x_0 \\ c\log(x) + d & \text{if } x_0 < x < 2 \end{cases}$$

The boundary conditions give $b = 0$ and $d = -c\log(2)$; as a result,

$$G(x,x_0) = \begin{cases} a\log(x) & \text{if } 1 < x < x_0 \\ c\log(x/2) & \text{if } x_0 < x < 2 \end{cases}$$

G must be continuous at x_0,

$$a\log(x_0) = c\log(x_0) - c\log(2)$$

and must satisfy the gap condition

$$-\int_{x_0-\epsilon}^{x_0+\epsilon} \partial_x(x\partial_x G(x,x_0))\mathsf{d}x = 1, \qquad \forall \epsilon > 0.$$

This gives

$$\begin{aligned} -x_0\left(\partial_x G(x_0^+,x_0) - G(x_0^-,x_0)\right) &= 1 \\ -x_0\left(\frac{c}{x_0} - \frac{a}{x_0}\right) &= 1 \end{aligned}$$

This gives

$$a - c = 1.$$

In conclusion $\log(x_0) = -c\log 2$ and

$$c = -\log(x_0)/\log(2), \quad a = 1 - \log(x_0)/\log(2) = \log(2/x_0)/\log(2).$$

This means

$$G(x,x_0) = \begin{cases} \frac{\log(2/x_0)}{\log(2)}\log(x) & \text{if } 1 < x < x_0 \\ \frac{\log(x_0)}{\log(2)}\log(2/x) & \text{if } x_0 < x < 2 \end{cases}$$

Question 105: Consider the equation $u'(x) + u = f(x)$ for $x \in (0,1)$ with $u(0) = a$. Let $G(x,x_0)$ be the associated Green's function. (Pay attention to the number of derivatives).
(a) Give the equation and boundary condition defining G and give an integral representation of $u(x_0)$ in terms of G, f and the boundary data a. (Do not compute G.)

Solution: The Green's function is defined by

$$-G'(x,x_0) + G(x,x_0) = \delta(x-x_0), \quad G(1,x_0) = 0.$$

We multiply the equation by u and we integrate over $(0,1)$ (in the distribution sense),

$$\int_0^1 -G'(x,x_0)u(x)\mathsf{d}x + \int_0^1 G(x,x_0)u(x)\mathsf{d}x = u(x_0).$$

We integrates by parts and we obtain,

$$u(x_0) = \int_0^1 G(x,x_0)(u'(x) + u(x))\mathsf{d}x - G(1,x_0)u(1) + G(0,x_0)u(0)$$

Then, using the fact that $u' + u = f$ and using the boundary conditions for G and u, we obtain

$$u(x_0) = \int_0^1 G(x, x_0) f(x) \mathsf{d}x + aG(0, x_0). \quad \forall x_0 \in (0, 1).$$

(b) Compute $G(x, x_0)$.

Solution: For $x < x_0$ and $x_0 > x$ we have

$$-G'(x, x_0) + G(x, x_0) = 0.$$

The solution is

$$G(x, x_0) = \begin{cases} \alpha e^x & \text{for } x < x_0 \\ \beta e^x & \text{for } x > x_0. \end{cases}$$

The boundary condition $G(1, x_0) = 0$ implies $\beta = 0$.

For every $\epsilon > 0$ we have

$$\begin{aligned} 1 &= \int_{x_0-\epsilon}^{x_0+\epsilon} (-G'(x, x_0) + G(x, x_0)) \mathsf{d}x \\ &= G(x_0 - \epsilon, x_0) - G(x_0 + \epsilon, x_0) + \int_{x_0-\epsilon}^{x_0+\epsilon} G(x, x_0) \mathsf{d}x \end{aligned}$$

The term $R_\epsilon = \int_{x_0-\epsilon}^{x_0+\epsilon} G(x, x_0) \mathsf{d}x$ can be bounded as follows:

$$|R_\epsilon| \leq 2\epsilon \max_{x \in [0,1]} |G(x, x_0)| = 2\epsilon\alpha e^{x_0}.$$

Clearly R_ϵ goes to 0 with ϵ. As a result we obtain the jump condition

$$1 = G(x_0^-, x_0) - G(x_0^+, x_0) = \alpha e^{x_0}.$$

This implies

$$\alpha = e^{-x_0}.$$

Finally

$$G(x, x_0) = \begin{cases} e^{x-x_0} & \text{for } x < x_0 \\ 0 & \text{for } x > x_0. \end{cases}$$

Question 106: Consider the equation $-\partial_x(x\partial_x u(x)) = f(x)$ for all $x \in (1, 2)$ with $u(1) = a$ and $u(2) = b$. Let $G(x, x_0)$ be the associated Green's function.
(i) Give the equation and boundary conditions satisfied by G and give the integral representation of $u(x_0)$ for all $x_0 \in (1, 2)$ in terms of G, f, and the boundary data. (Do not compute G in this question).

Solution: We have a second-order PDE and the operator is clearly self-adjoint. The Green's function solves the equation

$$-\partial_x(x\partial_x G(x, x_0)) = \delta(x - x_0), \quad G(1, x_0) = 0, \quad G(2, x_0) = 0.$$

We multiply the equation by u and integrate over the domain $(1, 2)$ (in the distribution sense).

$$\langle \delta(x - x_0), u \rangle = u(x_0) = -\int_1^2 \partial_x(x\partial_x G(x, x_0)) u(x) \mathsf{d}x.$$

We integrate by parts and we obtain,

$$\begin{aligned} u(x_0) &= \int_1^2 x\partial_x G(x, x_0) \partial_x u(x) \mathsf{d}x - [x\partial_x G(x, x_0) u(x)]_1^2 \\ &= -\int_1^2 G(x, x_0) \partial_x(x\partial_x u(x)) \mathsf{d}x - 2\partial_x G(2, x_0) u(2) + \partial_x G(1, x_0) u(1). \end{aligned}$$

Now, using the boundary conditions and the fact that $-\partial_x(x\partial_x u(x)) = f(x)$, we finally have

$$u(x_0) = \int_1^2 G(x,x_0)f(x)\mathsf{d}x - 2\partial_x G(2,x_0)b + \partial_x G(1,x_0)a.$$

(ii) Compute $G(x,x_0)$ for all $x, x_0 \in (1,2)$.

Solution: For all $x \neq x_0$ we have

$$-\partial_x(x\partial_x G(x,x_0)) = 0.$$

The solution is

$$G(x,x_0) = \begin{cases} a\log(x) + b & \text{if } 1 < x < x_0 \\ c\log(x) + d & \text{if } x_0 < x < 2 \end{cases}$$

The boundary conditions give $b = 0$ and $d = -c\log(2)$; as a result,

$$G(x,x_0) = \begin{cases} a\log(x) & \text{if } 1 < x < x_0 \\ c\log(x/2) & \text{if } x_0 < x < 2 \end{cases}$$

G must be continuous at x_0,

$$a\log(x_0) = c\log(x_0) - c\log(2)$$

and must satisfy the gap condition

$$-\int_{x_0-\epsilon}^{x_0+\epsilon} \partial_x(x\partial_x G(x,x_0))\mathsf{d}x = 1, \qquad \forall\epsilon > 0.$$

This gives

$$-x_0\left(\partial_x G(x_0^+,x_0) - G(x_0^-,x_0)\right) = 1$$
$$-x_0\left(\frac{c}{x_0} - \frac{a}{x_0}\right) = 1$$

This gives

$$a - c = 1.$$

In conclusion $\log(x_0) = -c\log 2$ and

$$c = -\log(x_0)/\log(2), \quad a = 1 - \log(x_0)/\log(2) = \log(2/x_0)/\log(2).$$

This means

$$G(x,x_0) = \begin{cases} \frac{\log(2/x_0)}{\log(2)}\log(x) & \text{if } 1 < x < x_0 \\ \frac{\log(x_0)}{\log(2)}\log(2/x) & \text{if } x_0 < x < 2 \end{cases}$$

Question 107: Consider the equation $\partial_{xx}u(x) = f(x)$, $x \in (0,L)$, with $u(0) = a$ and $\partial_x u(L) = b$.
(a) Compute the Green's function of the problem.

Solution: Let x_0 be a point in $(0,L)$. The Green's function of the problem is such that

$$\partial_{xx}G(x,x_0) = \delta_{x_0}, \quad G(0,x_0) = 0, \quad \partial_x G(L,x_0) = 0.$$

The following holds for all $x \in (0,x_0)$:

$$\partial_{xx}G(x,x_0) = 0.$$

This implies that $G(x,x_0) = ax + b$ in $(0,x_0)$. The boundary condition $G(0,x_0) = 0$ gives $b = 0$. Likewise, the following holds for all $x \in (x_0,L)$:

$$\partial_{xx}G(x,x_0) = 0.$$

This implies that $G(x,x_0) = cx + d$ in (x_0, L). The boundary condition $\partial_x G(L,x_0) = 0$ gives $c = 0$. The continuity of $G(x,x_0)$ at x_0 implies that $ax_0 = d$. The condition

$$\int_{-\epsilon}^{\epsilon} \partial_{xx} G(x,x_0)\mathsf{d}x = 1, \qquad \forall \epsilon > 0,$$

gives the so-called jump condition: $\partial_x G(x_0^+, x_0) - \partial_x G(x_0^-, x_0) = 1$. This means that $0 - a = 1$, i.e., $a = -1$ and $d = -x_0$. In conclusion

$$G(x,x_0) = \begin{cases} -x & \text{if} \le x \le x_0, \\ -x_0 & \text{otherwise.} \end{cases}$$

(b) Give the integral representation of u using the Green's function.

Solution: Let x_0 be a point in $(0, L)$. The definition of the Dirac measure at x_0 is such that

$$\begin{aligned}
u(x_0) &= \langle \delta_{x_0}, u \rangle = \langle \partial_{xx} G(\cdot, x_0), u \rangle \\
&= -\int_0^L \partial_x G(x,x_0)\partial_x u(x)\mathsf{d}x + [\partial_x G(x,x_0)u(x)]_0^L \\
&= \int_0^L G(x,x_0)\partial_{xx} u(x)\mathsf{d}x - [G(x,x_0)\partial_x u(x)]_0^L + [\partial_x G(x,x_0)u(x)]_0^L \\
&= \int_0^L G(x,x_0) f(x)\mathsf{d}x - G(L,x_0)\partial_x u(L) + G(0,x_0)\partial_x u(0) + \partial_x G(L,x_0)u(L) - \partial_x G(0,x_0)u(0).
\end{aligned}$$

This finally gives the following representation of the solution:

$$u(x_0) = \int_0^L G(x,x_0) f(x)\mathsf{d}x - G(L,x_0)b - \partial_x G(0,x_0)a$$
