1 Laplace equation

Question 1: (a) Find a function $U(x, y) = a + bx + cy + dxy$, such that $U(x, 0) = x$, $U(1, y) = 1 + y$, $U(x, 1) = 3x - 1$, and $U(0, y) = -y$.

Solution:

$$U(x, y) = x - y + 2xy$$

solves the problem.

(b) Use (a) to solve the PDE $u_{xx} + u_{yy} = 0$, $\forall(x, y) \in (0, 1)\times(0, 1)$, with the boundary conditions $u(x, 0) = 3\sin(\pi x) + x$, $u(1, y) = 1 + y$, $u(x, 1) = 3x - 1$,and $u(0, y) = \sin(2\pi y) - y$.

Solution: By setting $\phi = u - U$, we observe that $\phi_{xx} + \phi_{yy} = 0$ and at the boundary of the domain we have

$$\phi(x, 0) = 3\sin(\pi x), \quad \phi(1, y) = 0, \quad \phi(x, 1) = 0, \quad \text{and} \quad \phi(0, y) = \sin(2\pi y).$$

It is clear that

$$\phi(x, y) = 3\sin(\pi x)\frac{\sinh(\pi(1 - y))}{\sinh(\pi)} + \frac{\sinh(2\pi(1 - x))}{\sinh(2\pi)}\sin(2\pi y)$$

Then,

$$u(x, y) = \phi(x, y) + U(x, y)$$

Question 2: Consider the Laplace equation $\Delta u = 0$ in the rectangle $x \in [0, L]$, $y \in [0, H]$ with the boundary conditions $u(0, y) = 0$, $u(L, y) = 0$, $u(x, 0) = 0$, $u(x, H) = f(x)$.

(a) Is there any compatibility condition that f must satisfy for a smooth solution to exist?

Solution: f must be such that $f(0) = 0$ and $f(L) = 0$, otherwise u would not be continuous at the two upper corners of the domain.

(b) Solve the Equation.

Solution: Use the separation of variable technique. Let $u(x) = \phi(x)\psi(y)$. Then, provided ψ and ϕ are non zero functions, this implies $\frac{\phi''}{\phi} = -\frac{\psi''}{\psi} = \lambda$. Observe that $\phi(0) = \phi(L) = 0$. The usual energy technique implies that λ is negative. That is to say $\phi(x) = a\cos(\sqrt{\lambda}x) + b\sin(\sqrt{\lambda}x)$. The boundary conditions imply $a = 0$ and $\sqrt{\lambda}L = n\pi$, i.e., $\phi(x) = b\sin(n\pi x/L)$. The fact that λ is negative implies $\psi(y) = c\cosh(\sqrt{\lambda}y) + d\sinh(\sqrt{\lambda}y)$. The boundary condition at $y = 0$ implies $c = 0$. Then

$$u(xy) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi x}{L})\sinh(\frac{n\pi y}{L})$$

Question 3: Solve the PDE (note that the width and the height of the rectangle are not equal)

$$\begin{aligned}
&\partial_{xx}u + \partial_{yy}u = 0, && 0 < x < 1,\ 0 < y < 2,\\
&u(x, 0) = 8\sin(9\pi x),\ u(x, 2) = 0, && 0 < x < 1,\\
&u(0, y) = \sin(2\pi y), u(1, y) = 0, && 0 < y < 2.
\end{aligned}$$

Solution: The method of separation of variables tells us that the solution is a sum of terms like $\sin(n\pi x)\sinh(n\pi(y - 2))$ and $\sin(m\pi y/2)\sinh(m\pi(x - 1)/2)$. By looking at the boundary conditions we infer that there are two nonzero terms in the expansion: one corresponding to $n = 9$ and one corresponding to $m = 4$. This gives

$$u(x, y) = 8\sin(9\pi x)\frac{\sinh(9\pi(2 - y))}{\sinh(18\pi))} + \sin(2\pi y)\frac{\sinh(2\pi(1 - x))}{\sinh(2\pi))}$$

Question 4: Consider the Laplace equation $\Delta u = 0$ in the rectangle $x \in [0, L]$, $y \in [0, H]$ with the boundary conditions $u(0, y) = 0$, $u(L, y) = 0$, $u(x, 0) = 0$, $u(x, H) = f(x)$.

(a) Is there any compatibility condition that f must satisfy for a smooth solution to exist?

Solution: f must be such that $f(0) = 0$ and $f(L) = 0$, otherwise u would not be continuous at the two upper corners of the domain.

(b) Solve the Equation.

Solution: Use the separation of variable technique. Let $u(x) = \phi(x)\psi(y)$. Then, provided ψ and ϕ are non zero functions, this implies $\frac{\phi''}{\phi} = -\frac{\psi''}{\psi} = \lambda$. Observe that $\phi(0) = \phi(L) = 0$. The usual technique implies that λ is negative. That is to say $\phi(x) = a\cos(\sqrt{\lambda}x) + b\sin(\sqrt{\lambda}x)$. The boundary conditions imply $a = 0$ and $\sqrt{\lambda}L = n\pi$, i.e., $\phi(x) = b\sin(n\pi x/L)$. The fact that λ is negative implies $\psi(y) = c\cosh(\sqrt{\lambda}y) + d\sinh(\sqrt{\lambda}y)$. The boundary condition at $y = 0$ implies $c = 0$. Then the ansatz is

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi x}{L})\sinh(\frac{n\pi y}{L}),$$

and the usual computation gives

$$A_n = \frac{2}{L\sinh(\frac{n\pi H}{L})}\int_0^L f(\xi)\sin(\frac{n\pi\xi}{L})d\xi.$$

Question 5: Consider the Laplace equation $\Delta u = 0$ in the rectangle $x \in [0, L]$, $y \in [0, H]$ with the boundary conditions $u(0, y) = 0$, $\partial_x u(L, y) = 0$, $u(x, 0) = 0$, $u(x, H) = \sin(\frac{3}{2}\pi x/L)$. Solve the Equation using the method of separation of variables. (Give all the details.)

Solution: Let $u(x) = \phi(x)\psi(y)$. Then, provided ψ and ϕ are non zero functions, this implies $\frac{\phi''}{\phi} = -\frac{\psi''}{\psi} = \lambda$. Observe that $\phi(0) = 0$ and $\phi'(L) = 0$. The usual energy technique implies that λ is negative. That is to say $\phi(x) = a\cos(\sqrt{\lambda}x) + b\sin(\sqrt{\lambda}x)$. The Dirichlet condition at $x = 0$ implies $a = 0$. The Neumann condition at L implies $\cos(\sqrt{\lambda}L) = 0$, which implies $\sqrt{\lambda}L = (n+\frac{1}{2})\pi$, where n is an integer. This means that $\phi(x) = b\sin((n + \frac{1}{2})\pi x/L)$. The fact that λ is negative implies $\psi(y) = c\cosh(\sqrt{\lambda}y) + d\sinh(\sqrt{\lambda}y)$. The boundary condition at $y = 0$ implies $c = 0$. Then

$$u(x,y) = A\sin((n + \tfrac{1}{2})\pi\frac{x}{L})\sinh((n + \tfrac{1}{2})\pi\frac{y}{L}).$$

The boundary condition at $y = H$ gives

$$\sin(\frac{3}{2}\pi\frac{x}{L}) = A\sin((n + \tfrac{1}{2})\pi\frac{x}{L})\sinh((n + \tfrac{1}{2})\pi\frac{H}{L}),$$

which implies $n = 1$ and $A = \sinh^{-1}\left(\frac{\frac{3}{2}\pi H}{L}\right)$, i.e.,

$$u(x,y) = \frac{\sinh\left(\frac{\frac{3}{2}\pi y}{L}\right)}{\sinh\left(\frac{\frac{3}{2}\pi H}{L}\right)}\sin(\frac{3}{2}\pi x/L)$$

Question 6: Consider the equation $-\Delta u = 0$ in the rectangle $\{(x, y);\ x \in [0, L], y \in [0, H]\}$ with the boundary conditions $u(0, y) = 0$, $u(L, y) = -5\cos(\frac{3}{2}\pi\frac{y}{H})$, $\partial_y u(x, 0) = 0$, $u(x, H) = 0$. Solve the equation using the method of separation of variables. (Give all the details.)

Solution: Let $u(x) = \phi(x)\psi(y)$. Then, provided ψ and ϕ are non zero functions, this implies $\frac{\phi''(x)}{\phi(x)} = -\frac{\psi''(y)}{\psi(y)} = \lambda$. Observe that $\psi'(0) = 0$ and $\psi(H) = 0$. The energy technique applied to $-\psi''(y) = \lambda\psi(y)$ gives

$$\int_0^H -\psi''(y)\psi(y)dy = \int_0^H \psi'(y)^2dy - \psi'(H)\psi(H) + \psi'(0)\psi(0) = \lambda\int_0^H \psi(y)^2y,$$

which implies $\int_0^H \psi'(y)^2dy = \lambda\int_0^H \psi(y)^2dy$ since $\psi'(0) = 0$ and $\psi(H) = 0$. This in turn implies that λ is nonnegative. Actually λ cannot be zero since it would mean that $\psi = 0$, which would contradict the fact that the solution u is nonzero ($\lambda = 0 \Rightarrow \psi'(y) = 0 \Rightarrow \psi(y) = \psi(H) = 0$ for all $y \in [0, H]$). As a result λ is positive and

$$\psi(y) = a\cos(\sqrt{\lambda}y) + b\sin(\sqrt{\lambda}y).$$

The Neumann condition at $y = 0$ gives $b = 0$. The Dirichlet condition at H implies $\cos(\sqrt{\lambda}H) = 0$, which implies $\sqrt{\lambda}H = (n+\frac{1}{2})\pi$, where n is any integer. This means that $\psi(y) = a\cos((n+\frac{1}{2})\pi\frac{y}{H})$. The fact that λ is positive implies $\phi(x) = c\cosh(\sqrt{\lambda}x) + d\sinh(\sqrt{\lambda}x)$. The boundary condition at $x = 0$ implies $c = 0$. Then

$$u(x,y) = A\cos((n+\tfrac{1}{2})\pi\frac{y}{H})\sinh((n+\tfrac{1}{2})\pi\frac{x}{H}).$$

The boundary condition at $x = L$ gives

$$-5\cos(\frac{3}{2}\pi\frac{y}{H}) = A\cos((n+\tfrac{1}{2})\pi\frac{y}{H})\sinh((n+\tfrac{1}{2})\pi\frac{L}{H}),$$

which, by identification, implies $1 = 2$ and $A = -5\sinh^{-1}\left(\frac{\frac{3}{2}\pi L}{H}\right)$, i.e.,

$$u(x,y) = -5\frac{\sinh\left(\frac{\frac{3}{2}\pi x}{H}\right)}{\sinh\left(\frac{\frac{3}{2}\pi L}{H}\right)}\cos(\frac{3}{2}\pi\frac{y}{H}).$$

1.1 Cylindrical coordinates

Question 7: Using cylindrical coordinates and the method of separation of variables, solve the Laplace equation, $\frac{1}{r}\partial_r(r\partial_r u) + \frac{1}{r^2}\partial_{\theta\theta}u = 0$, inside the domain $D = \{\theta \in [0,\frac{\pi}{2}],\ r \in [0,1]\}$, subject to the boundary conditions $\partial_\theta u(r,0) = 0$, $u(r,\frac{\pi}{2}) = 0$, $u(1,\theta) = \cos(3\theta)$.

Solution: We set $u(r,\theta) = \phi(\theta)g(r)$. This means $\phi'' = -\lambda\phi$, with $\phi'(0) = 0$ and $\phi(\frac{\pi}{2}) = 0$, and $rd_r(rd_rg(r)) = \lambda g(r)$. Then using integration by parts plus the boundary conditions we prove that λ is non-negative. Then

$$\phi(\theta) = c_1\cos(\sqrt{\lambda}\theta) + c_2\sin(\sqrt{\lambda}\theta).$$

The boundary condition $\phi'(0) = 0$ implies $c_2 = 0$. The boundary condition $\phi(\frac{\pi}{2}) = 0$ implies $\sqrt{\lambda}\frac{\pi}{2} = (2n+1)\frac{\pi}{2}$ with $n \in \mathbb{N}$. This means $\sqrt{\lambda} = (2n+1)$. From class we know that $g(r)$ is of the form r^α, $\alpha \geq 0$. The equality $rd_r(rd_rr^\alpha) = \lambda r^\alpha$ gives $\alpha^2 = \lambda$. The condition $\alpha \geq 0$ implies $2n+1 = \alpha$. The boundary condition at $r = 1$ gives $\cos(3\theta) = 1^{2n+1}\cos((2n+1)\theta)$. This implies $n = 1$. The solution to the problem is

$$u(r,\theta) = r^3\cos(3\theta).$$

Question 8: Using cylindrical coordinates and the method of separation of variables, solve the Laplace equation, $\frac{1}{r}\partial_r(r\partial_r u) + \frac{1}{r^2}\partial_{\theta\theta}u = 0$, inside the domain $D = \{\theta \in [0,\frac{\pi}{2}],\ r \in [0,1]\}$, subject to the boundary conditions $u(r,0) = 0$, $u(r,\frac{\pi}{2}) = 0$, $u(1,\theta) = \sin(2\theta)$. (Give all the details.)

Solution: We set $u(r,\theta) = \phi(\theta)g(r)$. This means $\phi'' = -\lambda\phi$, with $\phi(0) = 0$ and $\phi(\frac{\pi}{2}) = 0$, and $rd_r(rd_rg(r)) = \lambda g(r)$. The usual energy argument applied to the two-point boundary value problem

$$\phi'' = -\lambda\phi, \qquad \phi(0) = 0, \qquad \phi(\frac{\pi}{2}) = 0,$$

implies that λ is non-negative. If $\lambda = 0$, then $\phi(\theta) = c_1 + c_2\theta$ and the boundary conditions imply $c_1 = c_2 = 0$, i.e., $\phi = 0$, which in turns gives $u = 0$ and this solution is incompatible with the boundary condition $u(1,\theta) = \sin(2\theta)$. Hence $\lambda > 0$ and

$$\phi(\theta) = c_1\cos(\sqrt{\lambda}\theta) + c_2\sin(\sqrt{\lambda}\theta).$$

The boundary condition $\phi(0) = 0$ implies $c_1 = 0$. The boundary condition $\phi(\frac{\pi}{2}) = 0$ implies $\sqrt{\lambda}\frac{\pi}{2} = n\pi$ with $n \in \mathbb{N}$. This means $\sqrt{\lambda} = 2n$. From class we know that $g(r)$ is of the form r^α, $\alpha \geq 0$. The equality $rd_r(rd_rr^\alpha) = \lambda r^\alpha$ gives $\alpha^2 = \lambda$. The condition $\alpha \geq 0$ implies $2n = \alpha$. The boundary condition at $r = 1$ gives $\sin(2\theta) = 1^{2n}\sin(2n\theta)$ for all $\theta \in [0, \frac{\pi}{2}]$. This implies $n = 1$. The solution to the problem is

$$u(r,\theta) = r^2 \sin(2\theta).$$

Question 9: Using cylindrical coordinates and the method of separation of variables, solve the equation, $\frac{1}{r}\partial_r(r\partial_r u) + \frac{1}{r^2}\partial_{\theta\theta}u = 0$, inside the domain $D = \{\theta \in [0,\pi],\ r \in [0,1]\}$, subject to the boundary conditions $u(r,0) = 0$, $u(r,\pi) = 0$, $u(1,\theta) = 2\sin(5\theta)$. (Give all the details.)

Solution: (1) We set $u(r,\theta) = \phi(\theta)g(r)$. This means $\phi'' = -\lambda\phi$, with $\phi(0) = 0$ and $\phi(\pi) = 0$, and $r\frac{\mathrm{d}}{\mathrm{d}r}(r\frac{\mathrm{d}}{\mathrm{d}r}g(r)) = \lambda g(r)$.

(2) The usual energy argument applied to the two-point boundary value problem

$$\phi'' = -\lambda\phi, \qquad \phi(0) = 0, \qquad \phi(\pi) = 0,$$

implies that λ is non-negative. If $\lambda = 0$, then $\phi(\theta) = c_1 + c_2\theta$ and the boundary conditions imply $c_1 = c_2 = 0$, i.e., $\phi = 0$, which in turns gives $u = 0$ and this solution is incompatible with the boundary condition $u(1,\theta) = 2\sin(5\theta)$. Hence $\lambda > 0$ and

$$\phi(\theta) = c_1\cos(\sqrt{\lambda}\theta) + c_2\sin(\sqrt{\lambda}\theta).$$

(3) The boundary condition $\phi(0) = 0$ implies $c_1 = 0$. The boundary condition $\phi(\pi) = 0$ implies $\sqrt{\lambda}\pi = n\pi$ with $n \in \mathbb{N}\setminus\{0\}$. This means $\sqrt{\lambda} = n$, $n = 1, 2, \ldots$.

(4) From class we know that $g(r)$ is of the form r^α, $\alpha \geq 0$. The equality $r\frac{\mathrm{d}}{\mathrm{d}r}(r\frac{\mathrm{d}}{\mathrm{d}r}r^\alpha) = \lambda r^\alpha$ gives $\alpha^2 = \lambda$. The condition $\alpha \geq 0$ implies $n = \alpha$. The boundary condition at $r = 1$ gives $2\sin(5\theta) = c_2 1^n \sin(n\theta)$ for all $\theta \in [0,\pi]$. This implies $n = 5$ and $c_2 = 2$.

(5) Finally, the solution to the problem is

$$u(r,\theta) = 2r^5\sin(5\theta).$$

1.2 Variable coefficients

Question 10: Let $k : [-1,+1] \longrightarrow \mathbb{R}$ be such that $k(x) = 1$, if $x \in [-1,0]$ and $k(x) = 2$ if $x \in (0,1]$. Solve the boundary value problem $-\partial_x(k(x)\partial_x T(x)) = 0$ with $\partial_x T(-1) = T(-1)$ and $\partial_x T(1) = 1$.
(i) What should be the interface conditions at $x = 0$ for this problem to make sense?

Solution: The function T and the flux $k(x)\partial_x T(x)$ must be continuous at $x = 0$. Let T^- denote the solution on $[-1,0]$ and T^+ the solution on $[0,+1]$. One should have $T^-(0) = T^+(0)$ and $k^-(0)\partial_x T^-(0) = k^+(0)\partial_x T^+(0)$, where $k^-(0) = 1$ and $k^+(0) = 2$.

(ii) Solve the problem, i.e., find $T(x)$, $x \in [-1,+1]$.

Solution: On $[-1,0]$ we have $k^-(x) = 1$, which implies $\partial_{xx}T^-(x) = 0$. This in turn implies $T^-(x) = a + bx$. The Robin boundary condition at $x = -1$ implies $\partial_x T^{-1}(-1) - T^-(-1) = 0 = 2b - a$. This gives $a = 2b$ and $T^-(x) = b(2+x)$.

We proceed similarly on $[0,+1]$ and we obtain $T^+(x) = c + dx$. The Neumann boundary condition at $x = +1$ gives $\partial_x T^+(+1) = 1 = d$ and $T^+(x) = c + x$.

The interface conditions $T^-(0) = T^+(0)$ and $k^-(0)\partial_x T^-(0) = k^+(0)\partial_x T^+(0)$ give

$$2b = c, \qquad \text{and} \qquad b = 2.$$

In conclusion

$$T(x) = \begin{cases} 2(2+x) & \text{if } x \in [-1,0], \\ 4+x & \text{if } x \in [0,+1]. \end{cases}$$

Question 11: Let $k : [-1,+1] \longrightarrow \mathbb{R}$ be such that $k(x) = 6$, if $x \in [-1, 0]$ and $k(x) = 3$ if $x \in (0, 1]$. Solve the boundary value problem $-\partial_x(k(x)\partial_x T(x)) = 0$ with $6\partial_x T(-1) = T(-1) + 13$ and $T(1) = 5$.
(i) What should be the interface conditions at $x = 0$ for this problem to make sense?

Solution: The function T and the flux $k(x)\partial_x T(x)$ must be continuous at $x = 0$. Let T^- denote the solution on $[-1, 0]$ and T^+ the solution on $[0,+1]$. One should have $T^-(0) = T^+(0)$ and $k^-(0)\partial_x T^-(0) = k^+(0)\partial_x T^+(0)$, where $k^-(0) = 6$ and $k^+(0) = 3$.

(ii) Solve the problem, i.e., find $T(x)$, $x \in [-1,+1]$.

Solution: On $[-1, 0]$ we have $k^-(x) = 1$, which implies $\partial_{xx} T^-(x) = 0$. This in turn implies $T^-(x) = a + bx$. The Robin boundary condition at $x = -1$ implies $6\partial_x T^-(-1) - T^-(-1) = 13 = 6b - (a - b)$. This gives $a = 7b - 13$ and $T^-(x) = 7b - 13 + bx$.

We proceed similarly on $[0,+1]$ and we obtain $T^+(x) = c + dx$. The Dirichlet boundary condition at $x = +1$ gives $T^+(1) = 5 = d + c$. This implies $c = 5 - d$ and $T^+(x) = 5 - d + dx$.

The interface conditions $T^-(0) = T^+(0)$ and $k^-(0)\partial_x T^-(0) = k^+(0)\partial_x T^+(0)$ give

$$7b - 13 = 5 - d, \qquad \text{and} \qquad 6b = 3d.$$

This implies $d = 4$ and $b = 2$. In conclusion

$$T(x) = \begin{cases} 2x + 1 & \text{if } x \in [-1, 0], \\ 4x + 1 & \text{if } x \in [0,+1]. \end{cases}$$

Question 12: Let $k : [-1,+1] \longrightarrow \mathbb{R}$ be such that $k(x) = 2$, if $x \in [-1, 0]$ and $k(x) = 3$ if $x \in (0, 1]$. Solve the boundary value problem $-\partial_x(k(x)\partial_x T(x)) = 0$ with $\partial_x T(-1) = T(-1)+3$ and $-\partial_x T(1) = T(1) - 7$.
(i) What should be the interface conditions at $x = 0$ for this problem to make sense?

Solution: The function T and the flux $k(x)\partial_x T(x)$ must be continuous at $x = 0$. Let T^- denote the solution on $[-1, 0]$ and T^+ the solution on $[0,+1]$. One should have $T^-(0) = T^+(0)$ and $k^-(0)\partial_x T^-(0) = k^+(0)\partial_x T^+(0)$, where $k^-(0) = 2$ and $k^+(0) = 3$.

(ii) Solve the problem, i.e., find $T(x)$, $x \in [-1,+1]$.

Solution: On $[-1, 0]$ we have $k^-(x) = 1$, which implies $\partial_{xx} T^-(x) = 0$. This in turn implies $T^-(x) = a + bx$. The Robin boundary condition at $x = -1$ implies $\partial_x T^-(-1) - T^-(-1) = 3 = 2b - a$. This gives $a = 2b - 3$ and $T^-(x) = 2b - 3 + bx$.

We proceed similarly on $[0,+1]$ and we obtain $T^+(x) = c + dx$. The Robin boundary condition at $x = +1$ gives $-\partial_x T^+(+1) - T^+(1) = -7 = -2d - c$. This implies $c = -2d + 7$ and $T^+(x) = -2d + 7 + dx$.

The interface conditions $T^-(0) = T^+(0)$ and $k^-(0)\partial_x T^-(0) = k^+(0)\partial_x T^+(0)$ give

$$2b - 3 = -2d + 7, \qquad \text{and} \qquad 2b = 3d.$$

This implies $d = 2$ and $b = 3$. In conclusion

$$T(x) = \begin{cases} 3x + 3 & \text{if } x \in [-1, 0], \\ 2x + 3 & \text{if } x \in [0,+1]. \end{cases}$$

1.3 Maximum principle

Question 13: Consider the square $D = (-1,+1)\times(-1,+1)$. Let $f(x, y) = x^2 - y^2 - 3$. Let $u \in \mathcal{C}^2(D) \cap \mathcal{C}^0(\overline{D})$ solve $-\Delta u = 0$ in D and $u|_{\partial D} = f$. Compute $\min_{(x,y)\in\overline{D}} u(x, y)$ and $\max_{(x,y)\in\overline{D}} u(x, y)$.

Solution: We use the maximum principle (u is harmonic and has the required regularity). Then

$$\min_{(x,y)\in\overline{D}} u(x,y) = \min_{(x,y)\in\partial D} f(x,y), \quad \text{and} \quad \max_{(x,y)\in\overline{D}} u(x,y) = \max_{(x,y)\in\partial D} f(x,y).$$

A point (x,y) is at the boundary of D if and only if $x^2 = 1$ and $y \in (-1,1)$ or $y^2 = 1$ and $x \in (-1,1)$. In the first case, $x^2 = 1$ and $y \in (-1,1)$, we have

$$f(x,y) = 1 - y^2 - 3, \qquad y \in (-1,1).$$

The maximum is -2 and the minimum is -3. In the second case, $y^2 = 1$ and $x \in (-1,1)$, we have

$$f(x,y) = x^2 - 1 - 3, \qquad x \in (-1,1).$$

The maximum is -3 and the minimum is -4. We finally can conclude

$$\min_{(x,y)\in\partial D} f(x,y) = \min_{-1\le x\le 1} x^2 - 4, = -4, \qquad \max_{(x,y)\in\partial D} f(x,y) = \max_{-1\le y\le 1} -2 - y^2 = -2.$$

In conclusion

$$\min_{(x,y)\in\overline{D}} u(x,y) = -4, \qquad \max_{(x,y)\in\overline{D}} u(x,y) = -2$$

Question 14: (i) Let Ω be an open connected set in $\mathbb{R}^2$. Let u be a real-valued nonconstant function continuous on $\overline{\Omega}$ and harmonic on Ω. Assume that there exists x_0 in Ω such $\nabla u(x_0) = 0$. Do we have a minimum, a maximum, or a saddle point at x_0? (explain)

Solution: The Maximum principle implies that u cannot be either minimum or maximum at x_0. This point is a saddle point.

(ii) Let $\Omega = (0,1)$ (note that $\overline{\Omega} = [0,1]$), and let $u : \overline{\Omega} \longrightarrow \mathbb{R}$ be such that $u(x) = 1$ for all $x \in \Omega$, $u(0) = 0$, and $u(1) = -1$. Is u harmonic on Ω? Find a point in $\overline{\Omega}$ where u reaches its maximum? Does this example contradict the Maximum Principle? (explain)

Solution: Yes u is harmonic on Ω since $u''(x) = 0$ for all x in Ω. Note however that u is not continuous on $\overline{\Omega}$; as a consequence, the hypotheses for the Maximum principle are not satisfied. In other words, the above example does not contradict the Maximum principle .

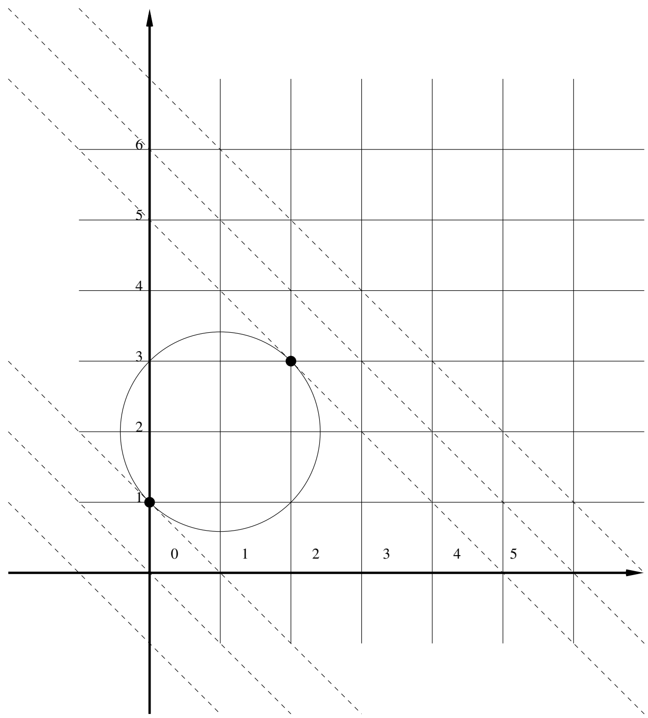
Question 15: Let u be a continuous function on $\overline{D}$ where D is some open, connected set in $\mathbb{R}^2$. Explain why, if u is harmonic, it is generally a waste of time to locate a point where u achieves its maximum by solving $\partial_x u = 0$ and $\partial_y u = 0$ simultaneously.

Solution: From the Maximum Principle, we know that if u is not constant, the maximum of u is achieved at the boundary of D. The zero gradient condition does not apply for maximums at the boundary.

Question 16: Let D be the open disk of radius $\sqrt{2}$ centered at $(1,2)$. Let $u \in \mathcal{C}^2(D) \cap \mathcal{C}^0(\overline{D})$ be a harmonic function in the disk D. Assume that $u(x,y) = (x+y)^2$ on the boundary of disk. Compute $\min_{(x,y)\in\overline{D}} u(x,y)$ and $\max_{(x,y)\in\overline{D}} u(x,y)$. You can give a geometric answer.

Solution: Since u is in $\mathcal{C}^2(D) \cap \mathcal{C}^0(\overline{D})$ and is harmonic, we can apply the maximum principle (Theorem). This theorem says that the maximum and minimum of u are attained at the boundary of the disk. The problem then amounts to finding the maximum and minimum of $(x+y)^2$ over the circle of radius $\sqrt{2}$ centered at $(1,2)$. The iso-values of the function $(x+y)^2$ are parallel lines of slope -1. These iso-line are perpendicular to the gradient of $(x+y)^2$ which is the vector $(2(x+y), 2(x+y)) = 2(x+y)(1,1)$. We must find the two tangent lines to the circle that are perpendicular to the vector $(1,1)$. One easily verify that $(0,1)$ and $(2,3)$ are the tangent points (verify that the segment connecting these two points is parallel to the vector $(1,1)$ and passes through the center of the circle). As a result

$$\min_{(x,y)\in\overline{D}} u(x,y) = 1 \qquad \text{and} \qquad \max_{(x,y)\in\overline{D}} u(x,y) = 25.$$



The dashed lines are iso-lines of $(x+y)^2$.

Question 17: Consider the disk D centered at $(0,0)$ of radius 1. Let $f(x,y) = x^2 - y^2 + 4y - 3$. Let $u \in \mathcal{C}^2(D) \cap \mathcal{C}^0(\overline{D})$ solve $-\Delta u = 0$ in D and $u|_{\partial D} = f$. Compute $\min_{(x,y)\in\overline{D}} u(x,y)$ and $\max_{(x,y)\in\overline{D}} u(x,y)$.

Solution: We use the maximum principle (u is harmonic and has the required regularity). Then

$$\min_{(x,y)\in\overline{D}} u(x,y) = \min_{(x,y)\in\partial D} f(x,y), \quad \text{and} \quad \max_{(x,y)\in\overline{D}} u(x,y) = \max_{(x,y)\in\partial D} f(x,y).$$

A point (x,y) is at the boundary of D if and only if $x^2+y^2 = 1$; as a result, the following holds for all $(x,y) \in \partial D$:

$$f(x,y) = x^2 - y^2 + 4y - 3 = 1 - y^2 - y^2 + 4y - 3 = -2(1-y)^2.$$

We obtain

$$\min_{(x,y)\in\partial D} f(x,y) = \min_{-1\le y\le 1} -2(1-y)^2 = -8, \qquad \max_{(x,y)\in\partial D} f(x,y) = \max_{-1\le y\le 1} -2(1-y)^2 = 0.$$

In conclusion

$$\min_{(x,y)\in\overline{D}} u(x,y) = -8, \qquad \max_{(x,y)\in\overline{D}} u(x,y) = 0$$

Question 18: Consider the square $D = (-1,+1)\times(-1,+1)$. Let $f(x,y) = x^2 - y^2 - 3$. Let $u \in \mathcal{C}^2(D) \cap \mathcal{C}^0(\overline{D})$ solve $-\Delta u = 0$ in D and $u|_{\partial D} = f$. Compute $\min_{(x,y)\in\overline{D}} u(x,y)$ and $\max_{(x,y)\in\overline{D}} u(x,y)$.

Solution: We use the maximum principle (u is harmonic and has the required regularity). Then

$$\min_{(x,y)\in\overline{D}} u(x,y) = \min_{(x,y)\in\partial D} f(x,y), \quad \text{and} \quad \max_{(x,y)\in\overline{D}} u(x,y) = \max_{(x,y)\in\partial D} f(x,y).$$

A point (x,y) is at the boundary of D if and only if $x^2 = 1$ and $y \in (-1,1)$ or $y^2 = 1$ and $x \in (-1,1)$. In the first case, $x^2 = 1$ and $y \in (-1,1)$, we have

$$f(x,y) = 1 - y^2 - 3, \qquad y \in (-1,1).$$

The maximum is -2 and the minimum is -3. In the second case, $y^2 = 1$ and $x \in (-1, 1)$, we have

$$f(x, y) = x^2 - 1 - 3, \qquad x \in (-1, 1).$$

The maximum is -3 and the minimum is -4. We finally can conclude

$$\min_{(x,y)\in\partial D} f(x, y) = \min_{-1\leq x\leq 1} x^2 - 4, = -4, \quad \max_{(x,y)\in\partial D} f(x, y) = \max_{-1\leq y\leq 1} -2 - y^2 = -2.$$

In conclusion

$$\min_{(x,y)\in\overline{D}} u(x, y) = -4, \quad \max_{(x,y)\in\overline{D}} u(x, y) = -2$$

2 Eigenvalue problems

Question 19: Consider the differential equation $\frac{d^2\phi}{dt^2} + \lambda\phi = 0$, $t \in (0, \pi)$, supplemented with the boundary conditions $\phi(0) = 0$, $\phi(\pi) = 0$.
(a) What is the sign of λ? Prove your answer.

Solution:

(b) Compute all the possible eigenvalues λ for this problem.

Solution:

Question 20: Let $\Omega = (0, L)$ and let (λ, u) be an eigenpair of the Laplace equation over Ω with zero Dirichlet condition. Assume that $\lambda \in \mathbb{C}$ and that the function $u(x)$ is complex-valued.
(i) Write the PDE solved by u.

Solution: u and λ are such

$$-\partial_{xx}u(x) = \lambda u(x), \quad u(0) = 0, \quad u(L) = 0.$$

(ii) Let $\bar{u}$ be the complex conjugate of u. Write the PDE solved by $\bar{u}$ (Hint: take the conjugate of (i)).

Solution: Taking the complex conjugate of (i) we obtain

$$-\partial_{xx}\bar{u}(x) = \overline{-\partial_{xx}u(x)} = \overline{\lambda u(x)} = \bar{\lambda}\bar{u}(x), \quad \bar{u}(0) = 0, \quad \bar{u}(L) = 0,$$

which gives

$$-\partial_{xx}\bar{u}(x) = \bar{\lambda}\bar{u}(x), \quad \bar{u}(0) = 0, \quad \bar{u}(L) = 0.$$

Note that we used the fact that $\overline{\partial_x u} = \partial_x \bar{u}$.

(iii) Prove that $\lambda \in \mathbb{R}$ (Hint: Use an energy argument with $\bar{u}$ in (i) and an energy argument with u (ii) and conclude that $\lambda = \bar{\lambda}$. Recall $u \neq 0$ and $|z|^2 = z\bar{z}$ for all $z \in \mathbb{C}$).

Solution: Multiply (i) by $\bar{u}$ and integrate over Ω

$$\begin{aligned}
\int_\Omega -\partial_{xx}u(x)\bar{u}(x)\mathrm{d}x &= \int_\Omega \lambda u(x)\bar{u}(x)\mathrm{d}x \\
\int_\Omega \partial_x u(x)\partial_x\bar{u}(x)\mathrm{d}x - [\partial_x u(x)\bar{u}(x)]_0^L &= \lambda \int_\Omega |u(x)|^2\mathrm{d}x \\
\int_\Omega \partial_x u(x)\overline{\partial_x u(x)}\mathrm{d}x &= \lambda \int_\Omega |u(x)|^2\mathrm{d}x \\
\int_\Omega |\partial_x u(x)|^2\mathrm{d}x &= \lambda \int_\Omega |u(x)|^2\mathrm{d}x.
\end{aligned}$$

Multiply (ii) by u and integrate over Ω

$$\begin{aligned}
\int_\Omega -\partial_{xx}\bar{u}(x)u(x)\mathrm{d}x &= \int_\Omega \bar{\lambda}\bar{u}(x)u(x)\mathrm{d}x \\
\int_\Omega \partial_x\bar{u}(x)\partial_x u(x)\mathrm{d}x - [\partial_x\bar{u}(x)u(x)]_0^L &= \bar{\lambda} \int_\Omega |u(x)|^2\mathrm{d}x \\
\int_\Omega \overline{\partial_x u(x)}\partial_x u(x)\mathrm{d}x &= \bar{\lambda} \int_\Omega |u(x)|^2\mathrm{d}x \\
\int_\Omega |\partial_x u(x)|^2\mathrm{d}x &= \bar{\lambda} \int_\Omega |u(x)|^2\mathrm{d}x.
\end{aligned}$$

In conclusion

$$\lambda \int_\Omega |u(x)|^2\mathrm{d}x = \bar{\lambda} \int_\Omega |u(x)|^2\mathrm{d}x,$$

which means

$$(\lambda - \bar{\lambda}) \int_\Omega |u(x)|^2\mathrm{d}x.$$

This in turn implies that $\lambda = \bar{\lambda}$ since $\int_\Omega |u(x)|^2 \mathrm{d}x$ is not zero (recall $u \neq 0$). In conclusion λ is real. Note in passing that this also prove that $\lambda \geq 0$.

Question 21: Consider the eigenvalue problem $-\frac{d}{dt}(t^{\frac{1}{2}}\frac{d}{dt}\phi(t)) = \lambda t^{-\frac{1}{2}}\phi(t)$, $t \in (0,1)$, supplemented with the boundary condition $\phi(0) = 0$, $\partial_t\phi(1) = 0$.
(a) Prove that it is necessary that λ be positive for a non-zero smooth solution to exist.

Solution: (i) Let ϕ be a non-zero smooth solution to the problem. Multiply the equation by ϕ and integrate over the domain. Use the Fundamental Theorem of calculus (i.e., integration by parts) to obtain

$$\int_0^1 t^{\frac{1}{2}}(\phi'(t))^2 \mathrm{d}t - [t^{\frac{1}{2}}\phi'(t)\phi(t)]_0^1 = \lambda \int_0^1 t^{-\frac{1}{2}}\phi^2(t)\mathrm{d}t.$$

Using the boundary conditions, we infer

$$\int_0^1 t^{\frac{1}{2}}(\phi'(t))^2 \mathrm{d}t = \lambda \int_0^1 t^{-\frac{1}{2}}\phi^2(t)\mathrm{d}t,$$

which means that λ is non-negative since ϕ is non-zero.

(ii) If $\lambda = 0$, then $\int_0^1 t^{\frac{1}{2}}(\phi'(t))^2\mathrm{d}t = 0$, which implies that $\phi'(t) = 0$ for all $t \in (0,1]$. This implies that $\phi(t)$ is constant, and this constant is zero since $\phi(0) = 0$. Hence, ϕ is zero if $\lambda = 0$. Since we want a nonzero solution, this implies that λ cannot be zero.

(iii) In conclusion, it is necessary that λ be positive for a nonzero smooth solution to exist.

(b) The general solution to $-\frac{d}{dt}(t^{\frac{1}{2}}\frac{d}{dt}\phi(t)) = \lambda t^{-\frac{1}{2}}\phi(t)$ is $\phi(t) = c_1\cos(2\sqrt{t}\sqrt{\lambda}) + c_2\sin(2\sqrt{t}\sqrt{\lambda})$ for $\lambda \geq 0$. Find all the eigenvalues $\lambda > 0$ and the associated nonzero eigenfunctions.

Solution: Since $\lambda \geq 0$ by hypothesis, ϕ is of the following form

$$\phi(t) = \phi(t) = c_1\cos(2\sqrt{t}\sqrt{\lambda}) + c_2\sin(2\sqrt{t}\sqrt{\lambda}).$$

The boundary condition $\phi(0) = 0$ implies $c_1 = 0$. The other boundary condition implies $\partial_x\phi(1) = 0 = c_2\sqrt{\lambda}\cos(2\sqrt{\lambda})$. The constant c_2 cannot be zero since we want ϕ to be nonzero; as a result, $2\sqrt{\lambda} = (n+\frac{1}{2})\pi$, $n = 0, 2, \ldots$. In conclusion

$$\lambda = ((2n+1)\pi)^2/16, \quad n = 1, 2, \ldots, \qquad \phi(t) = c\sin((n+\frac{1}{2})\pi\sqrt{t}).$$

Question 22: Let $\Omega = (0, L)$ and let (λ, u) be an eigenpair of the Laplace equation over Ω with zero Dirichlet condition. Assume that $\lambda \in \mathbb{C}$ and that the function $u(x)$ is complex-valued.
(i) Write the PDE solved by u.

Solution: u and λ are such

$$-\partial_{xx}u(x) = \lambda u(x), \quad u(0) = 0, \quad u(L) = 0.$$

(ii) Let $\bar{u}$ be the complex conjugate of u. Write the PDE solved by $\bar{u}$ (Hint: take the conjugate of (i)).

Solution: Take the complex conjugate of (i) we obtain

$$-\partial_{xx}\bar{u}(x) = \overline{-\partial_{xx}u(x)} = \overline{\lambda u(x)} = \bar{\lambda}\bar{u}(x), \quad \bar{u}(0) = 0, \quad \bar{u}(L) = 0$$

which gives

$$-\partial_{xx}\bar{u}(x) = \bar{\lambda}\bar{u}(x), \quad \bar{u}(0) = 0, \quad \bar{u}(L) = 0.$$

Note that we used the fact that $\overline{\partial_x u} = \partial_x\bar{u}$.

(iii) Prove that $\lambda \in \mathbb{R}$ (Hint: Use an energy argument with $\bar{u}$ in (i) and an energy argument with u (ii) and conclude that $\lambda = \bar{\lambda}$. Recall $u \neq 0$ and $|z|^2 = z\bar{z}$ for all $z \in \mathbb{C}$).

Solution: Multiply (i) by $\bar{u}$ and integrate over Ω

$$\int_\Omega -\partial_{xx}u(x)\bar{u}(x)\mathrm{d}x = \int_\Omega \lambda u(x)\bar{u}(x)\mathrm{d}x$$
$$\int_\Omega \partial_x u(x)\partial_x\bar{u}(x)\mathrm{d}x - [\partial_x u(x)\bar{u}(x)]_0^L = \lambda\int_\Omega |u(x)|^2\mathrm{d}x$$
$$\int_\Omega \partial_x u(x)\overline{\partial_x u(x)}\mathrm{d}x = \lambda\int_\Omega |u(x)|^2\mathrm{d}x$$
$$\int_\Omega |\partial_x u(x)|^2\mathrm{d}x = \lambda\int_\Omega |u(x)|^2\mathrm{d}x.$$

Multiply (ii) by u and integrate over Ω

$$\int_\Omega -\partial_{xx}\bar{u}(x)u(x)\mathrm{d}x = \int_\Omega \bar{\lambda}\bar{u}(x)u(x)\mathrm{d}x$$
$$\int_\Omega \partial_x\bar{u}(x)\partial_x u(x)\mathrm{d}x - [\partial_x\bar{u}(x)u(x)]_0^L = \bar{\lambda}\int_\Omega |u(x)|^2\mathrm{d}x$$
$$\int_\Omega \overline{\partial_x u(x)}\partial_x u(x)\mathrm{d}x = \bar{\lambda}\int_\Omega |u(x)|^2\mathrm{d}x$$
$$\int_\Omega |\partial_x u(x)|^2\mathrm{d}x = \bar{\lambda}\int_\Omega |u(x)|^2\mathrm{d}x.$$

In conclusion

$$\lambda\int_\Omega |u(x)|^2\mathrm{d}x = \bar{\lambda}\int_\Omega |u(x)|^2\mathrm{d}x,$$

which means

$$(\lambda-\bar{\lambda})\int_\Omega |u(x)|^2\mathrm{d}x.$$

This in turn implies that $\lambda=\bar{\lambda}$ since $\int_\Omega |u(x)|^2\mathrm{d}x$ is not zero (recall $u\neq 0$). In conclusion λ is real. Note in passing that this also prove that $\lambda\geq 0$.

Question 23: Consider the eigenvalue problem $-\frac{d}{dt}(t^{\frac{1}{2}}\frac{d}{dt}\phi(t)) = \lambda t^{-\frac{1}{2}}\phi(t)$, $t\in(0,1)$, supplemented with the boundary condition $\phi(0)=0$, $\phi(1)=0$.
(a) Prove that it is necessary that λ be positive for a non-zero smooth solution to exist.

Solution: (i) Let ϕ be a non-zero smooth solution to the problem. Multiply the equation by ϕ and integrate over the domain. Use the fundamental Theorem of calculus (i.e., integration by parts) to obtain

$$\int_0^1 t^{\frac{1}{2}}(\phi'(t))^2\mathrm{d}t - [t^{\frac{1}{2}}\phi'(t)\phi(t)]_0^1 = \lambda\int_0^1 t^{-\frac{1}{2}}\phi^2(t)\mathrm{d}t.$$

Using the boundary conditions, we infer

$$\int_0^1 t^{\frac{1}{2}}(\phi'(t))^2\mathrm{d}t = \lambda\int_0^1 t^{-\frac{1}{2}}\phi^2(t)\mathrm{d}t,$$

which means that λ is non-negative since ϕ is non-zero.

(ii) If $\lambda=0$, then $\int_0^1 t^{\frac{1}{2}}(\phi'(t))^2\mathrm{d}t = 0$, which implies that $\phi'(t)=0$ for all $t\in(0,1]$. The fundamental theorem of calculus applied between t and 1 implies $\phi(t)=\phi(1)+\int_1^t\phi'(\tau)d\tau=0$ since $\phi(1)=0$ and $\phi'(\tau)=0$ for all $\tau\in(t,1]$. Hence, ϕ is zero if $\lambda=0$. Since we want a nonzero solution, this implies that λ cannot be zero.

(iii) In conclusion, it is necessary that λ be positive for a nonzero smooth solution to exist.

(b) The general solution to $-\frac{d}{dt}(t^{\frac{1}{2}}\frac{d}{dt}\phi(t)) = \lambda t^{-\frac{1}{2}}\phi(t)$ is $\phi(t)=c_1\cos(2\sqrt{t}\sqrt{\lambda})+c_2\sin(2\sqrt{t}\sqrt{\lambda})$ for $\lambda\geq 0$. Find all the eigenvalues $\lambda\geq 0$ and the associated nonzero eigenfunctions.

Solution: Since $\lambda\geq 0$ by hypothesis, ϕ is of the following form

$$\phi(t)=\phi(t)=c_1\cos(2\sqrt{t}\sqrt{\lambda})+c_2\sin(2\sqrt{t}\sqrt{\lambda}).$$

The boundary condition $\phi(0) = 0$ implies $c_1 = 0$. The other boundary condition implies $\phi(1) = 0 = c_2 \sin(2\sqrt{\lambda})$. The constant c_2 cannot be zero since we want ϕ to be nonzero; as a result, $2\sqrt{\lambda} = n\pi$, $n = 1, 2, \ldots$. In conclusion

$$\lambda = (n\pi)^2/4, \quad n = 1, 2, \ldots, \qquad \phi(t) = c\sin(n\pi\sqrt{t}).$$

Question 24: Consider the differential equation $\frac{d^2\phi}{dt^2} + \lambda\phi = 0$, $t \in (0, \pi)$, supplemented with the boundary conditions $\phi(0) = 0$, $\phi'(\pi) = 0$.
(a) What is the sign of λ? Prove your answer.

Solution: Multiply the equation by ϕ and integrate over the domain.

$$-\int_0^\pi (\phi'(t))^2 dt + \phi'\phi|_0^\pi + \lambda \int_0^\pi \phi^2(t) dt = 0.$$

Using the BCs, we infer

$$-\int_0^\pi (\phi'(t))^2 dt = \lambda \int_0^\pi \phi^2(t) dt,$$

which means that λ is non-negative.

(b) Compute all the possible eigenvalues λ for this problem and compute ϕ.

Solution: There are two cases: Either $\lambda > 0$ or $\lambda = 0$. Assume first $\lambda > 0$, then

$$\phi(t) = c_1 \cos(\sqrt{\lambda}t) + c_2 \sin(\sqrt{\lambda}t).$$

The boundary condition $\phi(0) = 0$ implies imply $c_1 = 0$. The other BC implies $c_2 \cos(\sqrt{\lambda}\pi) = 0$. $c_2 = 0$ gives $\phi = 0$, which is not a proper eigenfunction. The other possibility is $\sqrt{\lambda}\pi = (n + \frac{1}{2})\pi$, $n \in \mathbb{N}$. In conclusion

$$\phi(t) = c_2 \sin((n + \tfrac{1}{2})t), \qquad n \in \mathbb{N}.$$

The case $\lambda = 0$ gives $\phi(t) = c_1 + c_2 t$. The BCs imply $c_1 = c_2 = 0$, i.e. $\phi = 0$, which is not a proper eigenfunction.

Question 25: Consider the eigenvalue problem $-\frac{d^2}{dt^2}\phi(t) + 2\frac{d}{dt}\phi(t) = \lambda\phi(t)$, $t \in (0, \pi)$, supplemented with the boundary condition $\phi(0) = 0$, $\phi(\pi) = 0$. (Hint: $2\phi(t)\frac{d}{dt}\phi(t) = \frac{d}{dt}\phi^2(t)$.)
(a) Prove that it is necessary that λ be positive for a non-zero solution to exist.

Solution: (i) Let ϕ be a non-zero solution to the problem. Multiply the equation by ϕ and integrate over the domain. Use the fundamental Theorem of calculus and use the hint to obtain

$$\int_0^\pi (\phi'(t))^2 dt - \phi'(\pi)\phi(\pi) + \phi'(0)\phi(0) + \int_0^\pi \frac{d}{dt}(\phi^2(t)) dt = \lambda \int_0^\pi \phi^2(t) dt$$
$$\int_0^\pi (\phi'(t))^2 dt - \phi'(\pi)\phi(\pi) + \phi'(0)\phi(0) + \phi^2(\pi) - \phi^2(0) = \lambda \int_0^\pi \phi^2(t) dt$$

Using the boundary conditions, we infer

$$\int_0^\pi (\phi'(t))^2 dt = \lambda \int_0^\pi \phi^2(t) dt,$$

which means that λ is non-negative since ϕ is non-zero.

(ii) If $\lambda = 0$, then $\int_0^\pi (\phi'(t))^2 dt = 0$ and $\phi(\pi)^2 = 0$, which implies that $\phi'(t) = 0$. The fundamental theorem of calculus implies $\phi(t) = \phi(0) + \int_0^t \phi'(\tau) d\tau = 0$. Hence, ϕ is zero if $\lambda = 0$. Since we want a nonzero solution, this implies that λ cannot be zero.

(iii) In conclusion, it is necessary that λ be positive for a nonzero solution to exist.

(b) The general solution to $-\phi'' + 2\phi' = \lambda\phi$ is $\phi(t) = e^t(c_1 \cos(t\sqrt{\lambda - 1}) + c_2 \sin(t\sqrt{\lambda - 1}))$ for $\lambda \geq 1$. Find all the eigenvalues $\lambda \geq 1$ and the associated nonzero eigenfunctions.

Solution: Since $\lambda \geq 1$ by hypothesis, ϕ is of the following form

$$\phi(t) = e^t(c_1 \cos(\sqrt{\lambda - 1}t) + c_2 \sin(\sqrt{\lambda - 1}t)).$$

The boundary condition $\phi(0) = 0$ implies $c_1 = 0$. The other boundary condition implies $\phi(\pi) = 0 = e^\pi c_2 \sin(\sqrt{\lambda - 1}\pi)$. The constant c_2 cannot be zero since we want ϕ to be nonzero; as a result, $\sqrt{\lambda - 1} = n$, $n = 1, 2, \ldots$. In conclusion

$$\lambda = n^2 + 1, \quad n = 1, 2, \ldots, \qquad \phi(t) = ce^t \sin(nt)$$

Question 26: Consider the eigenvalue problem $-\frac{d}{dt}(t^{\frac{1}{2}} \frac{d}{dt}\phi(t)) = \lambda t^{-\frac{1}{2}}\phi(t)$, $t \in (0, 1)$, supplemented with the boundary condition $\phi(0) = 0$, $\phi(1) = 0$.
(a) Prove that it is necessary that λ be positive for a non-zero smooth solution to exist.

Solution: (i) Let ϕ be a non-zero smooth solution to the problem. Multiply the equation by ϕ and integrate over the domain. Use the fundamental Theorem of calculus (i.e., integration by parts) to obtain

$$\int_0^1 t^{\frac{1}{2}}(\phi'(t))^2 \mathrm{d}t - [t^{\frac{1}{2}}\phi'(t)\phi(t)]_0^1 = \lambda \int_0^1 t^{-\frac{1}{2}}\phi^2(t)\mathrm{d}t.$$

Using the boundary conditions, we infer

$$\int_0^1 t^{\frac{1}{2}}(\phi'(t))^2 \mathrm{d}t = \lambda \int_0^1 t^{-\frac{1}{2}}\phi^2(t)\mathrm{d}t,$$

which means that λ is non-negative since ϕ is non-zero.

(ii) If $\lambda = 0$, then $\int_0^1 t^{\frac{1}{2}}(\phi'(t))^2 \mathrm{d}t = 0$, which implies that $\phi'(t) = 0$ for all $t \in (0, 1]$. The fundamental theorem of calculus applied between t and 1 implies $\phi(t) = \phi(1) + \int_1^t \phi'(\tau)d\tau = 0$ since $\phi(1) = 0$ and $\phi'(\tau) = 0$ for all $\tau \in (t, 1]$. Hence, ϕ is zero if $\lambda = 0$. Since we want a nonzero solution, this implies that λ cannot be zero.

(iii) In conclusion, it is necessary that λ be positive for a nonzero smooth solution to exist.

(b) The general solution to $-\frac{d}{dt}(t^{\frac{1}{2}} \frac{d}{dt}\phi(t)) = \lambda t^{-\frac{1}{2}}\phi(t)$ is $\phi(t) = c_1 \cos(2\sqrt{t}\sqrt{\lambda}) + c_2 \sin(2\sqrt{t}\sqrt{\lambda})$ for $\lambda \geq 0$. Find all the eigenvalues $\lambda \geq 0$ and the associated nonzero eigenfunctions.

Solution: Since $\lambda \geq 0$ by hypothesis, ϕ is of the following form

$$\phi(t) = \phi(t) = c_1 \cos(2\sqrt{t}\sqrt{\lambda}) + c_2 \sin(2\sqrt{t}\sqrt{\lambda}).$$

The boundary condition $\phi(0) = 0$ implies $c_1 = 0$. The other boundary condition implies $\phi(1) = 0 = c_2 \sin(2\sqrt{\lambda})$. The constant c_2 cannot be zero since we want ϕ to be nonzero; as a result, $2\sqrt{\lambda} = n\pi$, $n = 1, 2, \ldots$. In conclusion

$$\lambda = (n\pi)^2/4, \quad n = 1, 2, \ldots, \qquad \phi(t) = c \sin(n\pi\sqrt{t}).$$

Question 27: Consider the differential equation $-\frac{d^2\phi}{dt^2} = \lambda\phi$, $t \in (0, \pi)$, supplemented with the boundary conditions $2\phi(0) = \phi'(0)$, $\phi(\pi) = 0$.
(a) What should be the sign of λ for a non-zero solution to exist? Prove your answer.

Solution: Let ϕ be a non-zero solution to the problem. Multiply the equation by ϕ and integrate over the domain.

$$\int_0^\pi (\phi'(t))^2 dt - \phi'(\pi)\phi(\pi) + \phi'(0)\phi(0) = \lambda \int_0^\pi \phi^2(t)dt.$$

Using the BCs, we infer

$$\int_0^\pi (\phi'(t))^2 dt + 2\phi(0)^2 = \lambda \int_0^\pi \phi^2(t)dt,$$

which means that λ is non-negative since ϕ is non-zero.

(b) Assume that $2\sin(\sqrt{\lambda}\pi)+\sqrt{\lambda}\cos(\sqrt{\lambda}\pi) \neq 0$. Prove that $\phi = 0$ is the only possible solution.

Solution: Observe first that λ cannot be zero, otherwise we would have $2\sin(\sqrt{\lambda}\pi)+\sqrt{\lambda}\cos(\sqrt{\lambda}\pi) = 0$. As a result, (a) implies that λ is positive. Then ϕ is of the following form

$$\phi(t) = c_1\cos(\sqrt{\lambda}t) + c_2\sin(\sqrt{\lambda}t).$$

The boundary condition $\phi'(0) = 2\phi(0)$ implies $2c_1 - \sqrt{\lambda}c_2 = 0$. The other BC implies $c_1\cos(\sqrt{\lambda}\pi) + c_2\sin(\sqrt{\lambda}\pi) = 0$. The constants c_1 and c_2 solve the following linear system

$$\begin{cases} 2c_1 - \sqrt{\lambda}c_2 = 0 \\ c_1\cos(\sqrt{\lambda}\pi) + c_2\sin(\sqrt{\lambda}\pi) = 0 \end{cases}$$

The determinant is equal to $2\sin(\sqrt{\lambda}\pi) + \sqrt{\lambda}\cos(\sqrt{\lambda}\pi)$ and is non-zero by hypothesis. Then the only solution is $c_1 = c_2 = 0$, which again gives $\phi = 0$.

In conclusion, the only possible solution to the problem is $\phi = 0$ if $2\sin(\sqrt{\lambda}\pi)+\sqrt{\lambda}\cos(\sqrt{\lambda}\pi) \neq 0$.

3 Fourier Series

Here are some formulae that you may want to use:

$$FS(f)(x) = \sum_{n=0}^{+\infty} a_n \cos(n\pi x/L) + \sum_{n=1}^{+\infty} b_n \sin(n\pi x/L), \tag{1}$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\mathrm{d}x, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(n\pi\frac{x}{L})\mathrm{d}x, \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(n\pi\frac{x}{L})\mathrm{d}x \tag{2}$$

Question 28: Compute the complex Fourier series of the function $f(x) = x$ defined on $[-\pi, \pi]$.

Solution: By definition $FS(f)(x) = \sum_{-\infty}^{+\infty} c_n e^{-in\pi x/L}$ where L is half the size of the interval on which f is defined. Here $L = \pi$. Hence, by integrating by parts once we obtain

$$\begin{aligned} c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} xe^{-inx} &= -\frac{1}{2\pi}\frac{1}{-in}\int_{-\pi}^{\pi} e^{-inx} + \frac{1}{2\pi}\frac{1}{-in}(\pi e^{-in\pi} + \pi e^{in\pi}) \\ &= \frac{1}{2\pi}\frac{1}{-in}2\pi(-1)^n. \end{aligned}$$

That is $c_n = \frac{(-1)^{n-1}}{in}$ and

$$FS(f)(x) = \sum_{-\infty}^{+\infty} \frac{(-1)^{n-1}}{in} e^{-inx}$$

Question 29: Let $f(x) = x$, $x \in [-L, L]$.
(a) Sketch the graph of the Fourier series of f for $x \in (-\infty, \infty)$.

Solution: The Fourier series is equal to the periodic extension of f, except at the points nL, $n \in \mathbb{Z}$ where it is equal to $0 = \frac{1}{2}(1-1)$.

(b) Compute the Fourier series of f.

Solution: f is odd, hence the cosine coefficients are zero. The sign coefficients b_n are given by

$$b_n = \frac{1}{L}\int_{-L}^{L} x\sin(\frac{n\pi x}{L})dx = \frac{L}{n\pi}\frac{1}{L}\int_{-L}^{L}\cos(\frac{n\pi x}{L})dx - 2\frac{L}{n\pi}\cos(n\pi).$$

As a result $b_n = 2(-1)^{n+1}\frac{L}{n\pi}$ and

$$FS(f)(x) = \sum_{1}^{\infty} 2(-1)^n \frac{L}{n\pi}\sin(\frac{n\pi x}{L}).$$

Question 30: Let N be a positive integer and let $\mathbb{P}_N$ be the set of trigonometric polynomials of degree at most N; that is, $\mathbb{P}_N = \mathrm{span}\{1, \cos(x), \sin(x), \dots, \cos(Nx), \sin(Nx)\}$.

(i) Consider the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n^5}\sin(7n)\cos(2nx).$$

Compute the best L^2-approximation of f in $\mathbb{P}_7$ over $(0, 2\pi)$.

Solution: The best L^2-approximation of f in $\mathbb{P}_7$ over $(0, 2\pi)$ is the truncated Fourier series $FS_7(f)$. Clearly

$$FS_7(f)(x) = \sum_{n=0}^{3} \frac{1}{n^5}\sin(7n)\cos(2nx)$$

(ii) Consider the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$f(x) = \sum_{n=0}^{11} \frac{1}{n^3+1} \cos(35n^2) \sin(3nx).$$

Compute the best L^2-approximation of f in $\mathbb{P}_{35}$ over $(0, 2\pi)$.

Solution: The best L^2-approximation of f in $\mathbb{P}_{35}$ over $(0, 2\pi)$ is the truncated Fourier series $FS_{35}(f)$. Observing that $f \in \mathbb{P}_{33} \subset \mathbb{P}_{35}$ it is clear that $FS_{35}(f) = f$.

Question 31: Let $f(x) = x$, $x \in [-L, L]$.
(a) Sketch the graph of the Fourier series of f for $x \in (-\infty, \infty)$.

Solution: The Fourier series is equal to the periodic extension of f, except at the points $(2n+1)L$, $n \in \mathbb{Z}$ where it is equal to $0 = \frac{1}{2}(1-1)$.

(b) Compute the Fourier series of f.

Solution: f is odd, hence the cosine coefficients are zero. The sign coefficients b_n are given by

$$b_n = \frac{1}{L}\int_{-L}^{L} x \sin(\frac{n\pi x}{L})dx = \frac{L}{n\pi}\frac{1}{L}\int_{-L}^{L} \cos(\frac{n\pi x}{L})dx - 2\frac{L}{n\pi}\cos(n\pi).$$

As a result $b_n = -2\cos(n\pi)\frac{L}{n\pi} = 2(-1)^{n+1}\frac{L}{n\pi}$ and

$$FS(f)(x) = \sum_{1}^{\infty} 2(-1)^n \frac{L}{n\pi} \sin(\frac{n\pi x}{L}).$$

Question 32: Let L be a positive real number.
(a) Compute the Fourier series of the function $(-L, L) \ni x \longmapsto x^2$.

Solution: The Function is even; as a result its sine coefficients are zero. We compute the cosine coefficients as follows:

$$\int_{-L}^{+L} x^2 \cos(m\pi x/L)dx = a_m \int_{-L}^{+L} \cos(m\pi x/L)^2 dx,$$

If $m = 0$, $c_0 = \frac{1}{2L}\int_{-L}^{+L} x^2 dx = \frac{1}{3}L$. Otherwise,

$$a_m = \frac{1}{L}\int_{-L}^{+L} x^2 \cos(m\pi x/L)dx.$$

Integration by parts two times gives

$$\begin{aligned} a_m &= -\frac{1}{L}\frac{L}{m\pi}\int_{-L}^{+L} 2x\sin(m\pi x/L)dx \\ &= 2\frac{L}{m^2\pi^2} x\cos(m\pi x/L)|_{-L}^{L} = 4\frac{L^2}{m^2\pi^2}(-1)^m. \end{aligned}$$

(b) For which values of x is the Fourier series equal to x^2?

Solution: The periodic extension of x^2 over $\mathbb{R}$ is piecewise smooth and globally continuous. This means that the Fourier series is equal to x^2 over the entire interval $[-L, +L]$.

(c) Using (a) and (b) give the Fourier series of x over $[-L, +L]$ and say where it is equal to x.

Solution: Since we can differentiate cosine series, the Fourier series of x over $[-L, +L]$ is obtained by differentiating that of x^2,

$$FS(x)(x) = \frac{d}{dx}FS(\frac{1}{2}x^2)(x) = \sum_{1}^{\infty} 2(-1)^{n+1}\frac{L}{n\pi}\sin(\frac{n\pi x}{L}).$$

We have equality $x = FS(x)(x)$ only on $(-L, +L)$. At $-L$ and $+L$ the Fourier series is zero.

Question 33: Let $f : [-1, +1] \longrightarrow \mathbb{R}$ be such that $f(x) = x$, if $x \in [0, 1]$ and $f(x) = 0$ if $x \in [-1, 0]$.
(a) Sketch the graph of the Fourier series of f on $(-\infty, \infty)$.

Solution: The Fourier series is equal to the periodic extension of f, except at the points $(2n+1)L$, $n \in \mathbb{Z}$, where it is equal to $\frac{1}{2} = \frac{1}{2}(1 - 0)$.

(b) Compute the Fourier coefficients of f (recall $a_0 = \frac{1}{2}\int_{-1}^{1} f(x)dx$, and for $n \geq 1$, $a_n = \int_{-1}^{1} f(x)\cos(n\pi x)dx$, $b_n = \int_{-1}^{1} f(x)\sin(n\pi x)dx$. Hint: integrate by parts).

Solution: Clearly $a_0 = \frac{1}{4}$. For $n \geq 1$ we have

$$a_n = \int_0^1 x\cos(n\pi x)dx = -\frac{1}{n\pi}\int_0^1 \sin(n\pi x) + \frac{1}{n\pi}x\sin(n\pi x)|_0^1 = \frac{1}{(n\pi)^2}\cos(n\pi x)|_0^1 = \frac{(-1)^n - 1}{(n\pi)^2}$$

and

$$b_n = \int_0^1 x\sin(n\pi x)dx = \frac{1}{n\pi}\int_0^1 \cos(n\pi x) - \frac{1}{n\pi}x\cos(n\pi x)|_0^1 = \frac{(-1)^{n+1}}{n\pi}.$$

Question 34: Consider $f : [-L, L] \longrightarrow \mathbb{R}$, $f(x) = x^2$. (a) Sketch the graph of the Fourier series of f.

Solution: $FS(f)$ is equal to the periodic extension of $f(x)$ over $\mathbb{R}$.

(b) For which values of x is $FS(f)$ equal to x^2?

Solution: The periodic extension of $f(x) = x^2$ over $\mathbb{R}$ is piecewise smooth and globally continuous. This means that the Fourier series is equal to x^2 over the entire interval $[-L, +L]$.

(c) Is it possible to obtain $FS(x)$ by differentiating $\frac{1}{2}FS(f)$ term by term?

Solution: Yes it is possible since the periodic extension of $f(x) = x^2$ over $\mathbb{R}$ is continuous and piecewise smooth.

Question 35: Let $f(x) = x$, $x \in [-L, L]$. (a) Sketch the graph of the Fourier series of f.

Solution: The Fourier series is equal to the periodic extension of f, except at the points $(2n+1)L$, $n \in \mathbb{Z}$ where it is equal to $0 = \frac{1}{2}(1 - 1)$.

(b) Compute the coefficients of the Fourier series of f. (Hint: $\int_a^b tg(t)\mathrm{d}t = [t\int g]_a^b - \int_a^b(\int g)(t)\mathrm{d}t$).

Solution: f is odd, hence the cosine coefficients are zero. The sine coefficients b_n are obtained by integration by parts

$$b_n = \frac{1}{L}\int_{-L}^{L} x\sin(\frac{n\pi x}{L})dx = -\frac{1}{L}\frac{L}{n\pi}[x\cos(n\pi\frac{x}{L})]_{-L}^{+L} + \frac{L}{n\pi}\frac{1}{L}\int_{-L}^{L}\cos(\frac{n\pi x}{L})dx.$$

As a result $b_n = -2\cos(n\pi)\frac{L}{n\pi} = 2(-1)^{n+1}\frac{L}{n\pi}$ and

$$FS(f)(x) = \frac{2L}{\pi}\sum_{1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(\frac{n\pi x}{L}).$$

Question 36: Let L be a positive real number. Let $V = \text{span}\{1, \cos(\pi t/L), \sin(\pi t/L)\}$ and consider the norm $\|f\|_{L^2} = \left(\int_{-L}^{L} f(t)^2\mathrm{d}t\right)^{\frac{1}{2}}$. (a) Compute the best approximation of $1 + t$ in V with respect to the above norm.

Solution: We know from class that the truncated Fourier series

$$FS_1(t) = a_0 + a_1\cos(\pi t/L) + b_1\sin(\pi t/L)$$

is the best approximation. Now we compute a_0, a_1, a_2

$$a_0 = \frac{1}{2L}\int_{-L}^{L}(1+t)\mathsf{d}t = 1,$$
$$a_1 = \frac{1}{L}\int_{-L}^{L}(1+t)\cos(\pi t/L)\mathsf{d}t = 0$$
$$b_1 = \frac{1}{L}\int_{-L}^{L}(1+t)\sin(\pi t/L)\mathsf{d}t = \frac{1}{L}\int_{-L}^{L}t\sin(\pi t/L)\mathsf{d}t = -2\cos(\pi)\frac{L}{\pi} = \frac{2L}{\pi}.$$

As a result

$$FS_1(t) = 1 + \frac{2L}{\pi}\sin(\pi t/L)$$

(b) Compute the best approximation of $3 + 2\cos(\pi t/L) - 5\sin(\pi t/L)$ in V.

Solution: The function $3 + 2\cos(\pi t/L) - 5\sin(\pi t/L)$ is a member of V; as a result, The best approximation is the function itself.

Question 37: Consider $f : [-L, L] \longrightarrow \mathbb{R}$, $f(x) = x^4$. (a) Sketch the graph of the Fourier series of f.

Solution: $FS(f)$ is equal to the periodic extension of $f(x)$ over $\mathbb{R}$.

(b) For which values of $x \in \mathbb{R}$ is $FS(f)$ equal to x^4? (Explain)

Solution: The periodic extension of $f(x) = x^4$ over $\mathbb{R}$ is piecewise smooth and globally continuous since $f(L) = f(-L)$. This means that the Fourier series is equal to x^4 over the entire interval $[-L, +L]$.

(c) Is it possible to obtain $FS(x^3)$ by differentiating $\frac{1}{4}FS(x^4)$ term by term? Which values this legitimate? (Explain)

Solution: Yes it is possible since the periodic extension of $f(x) = x^4$ over $\mathbb{R}$ is continuous and piecewise smooth. This operation is legitimate everywhere the function $FS(x^3)$ is smooth, i.e., for all the points in $\mathbb{R}\backslash\{2k-1,\ k \in \mathbb{Z}\}$ (i.e., one needs to exclude the points $\ldots, -7, -5, -3, -1, +1, +3, +5+7, \ldots$

Question 38: Let L be a positive real number. Let $\mathbb{P}_1 = \text{span}\{1, \cos(\pi t/L), \sin(\pi t/L)\}$ and consider the norm $\|f\|_{L^2} = \left(\int_{-L}^{L} f(t)^2 \mathrm{d}t\right)^{\frac{1}{2}}$. (a) Compute the best approximation of $2 - 3t$ in $\mathbb{P}_1$ with respect to the above norm. (Hint: $\int_{-L}^{L} t\sin(\pi t/L)\mathrm{d}t = 2L^2/\pi$.)

Solution: We know from class that the truncated Fourier series

$$FS_1(t) = a_0 + a_1\cos(\pi t/L) + b_1\sin(\pi t/L)$$

is the best approximation. Now we compute a_0, a_1, a_2

$$a_0 = \frac{1}{2L}\int_{-L}^{L}(2-3t)\mathsf{d}t = 2,$$
$$a_1 = \frac{1}{L}\int_{-L}^{L}(2-3t)\cos(\pi t/L)\mathsf{d}t = 0$$
$$b_1 = \frac{1}{L}\int_{-L}^{L}(2-3t)\sin(\pi t/L)\mathsf{d}t = \frac{1}{L}\int_{-L}^{L}-3t\sin(\pi t/L)\mathsf{d}t = -6\cos(\pi)\frac{L}{\pi} = -\frac{6L}{\pi}.$$

As a result

$$FS_1(t) = 2 - \frac{6L}{\pi}\sin(\pi t/L)$$

(b) Compute the best approximation of $h(t) = 2\cos(\pi t/L) + 7\sin(3\pi t/L)$ in $\mathbb{P}_1$.

Solution: The function $h(t) - 2\cos(\pi t/L) = 7\sin(3\pi t/L)$ is orthogonal to all the members of $\mathbb{P}_1$ since the functions $\cos(m\pi t/L)$ and $\sin(m\pi t/L)$ are orthogonal to both $\cos(n\pi t/L)$ and $\sin(n\pi t/L)$

for all $m \neq m$; as a result, the best approximation of h in $\mathbb{P}_1$ is $2\cos(\pi t/L)$. (Recall that the best approximation of h in $\mathbb{P}_1$ is such that $\int_{-L}^{L}(h(t) - FS_1(h))p(t)\mathrm{d}t = 0$ for all $p \in \mathbb{P}_1$.) In conclusion

$$FS_1(h) = 2\cos(\pi t/L).$$

Question 39: Consider $f : [-L, L] \longrightarrow \mathbb{R}$, $f(x) = x^4$. (a) Sketch the graph of the Fourier series of f.

Solution: $FS(f)$ is equal to the periodic extension of $f(x)$ over $\mathbb{R}$.

(b) For which values of $x \in \mathbb{R}$ is $FS(f)$ equal to x^4? (Explain)

Solution: The periodic extension of $f(x) = x^4$ over $\mathbb{R}$ is piecewise smooth and globally continuous since $f(L) = f(-L)$. This means that the Fourier series is equal to x^4 over the entire interval $[-L, +L]$.

(c) Is it possible to obtain $FS(x^3)$ by differentiating $\frac{1}{4}FS(x^4)$ term by term? (Explain)

Solution: Yes it is possible since the periodic extension of $f(x) = x^4$ over $\mathbb{R}$ is continuous and piecewise smooth.

Question 40: Let L be a positive real number. Let $P_1 = \mathrm{span}\{1, \cos(\pi t/L), \sin(\pi t/L)\}$ and consider the norm $\|f\|_{L^2} = \left(\int_{-L}^{L} f(t)^2\mathrm{d}t\right)^{\frac{1}{2}}$. (a) Compute the best approximation of $1+t$ in V with respect to the above norm. (Hint: $\int_{-L}^{L} t\sin(\pi t/L)\mathrm{d}t = 2L^2/\pi$.)

Solution: We know from class that the truncated Fourier series

$$FS_1(t) = a_0 + a_1\cos(\pi t/L) + b_1\sin(\pi t/L)$$

is the best approximation. Now we compute a_0, a_1, a_2

$$a_0 = \frac{1}{2L}\int_{-L}^{L}(1+t)\mathrm{d}t = 1,$$
$$a_1 = \frac{1}{L}\int_{-L}^{L}(1+t)\cos(\pi t/L)\mathrm{d}t = 0$$
$$b_1 = \frac{1}{L}\int_{-L}^{L}(1+t)\sin(\pi t/L)\mathrm{d}t = \frac{1}{L}\int_{-L}^{L} t\sin(\pi t/L)\mathrm{d}t = -2\cos(\pi)\frac{L}{\pi} = \frac{2L}{\pi}.$$

As a result

$$FS_1(t) = 1 + \frac{2L}{\pi}\sin(\pi t/L)$$

(b) Compute the best approximation of $h(t) = 2\cos(2\pi t/L) - 5\sin(3\pi t/L)$ in P_1.

Solution: The function $h(t)2\cos(2\pi t/L) - 5\sin(3\pi t/L)$ is orthogonal to all the members of P_1 since the functions $\cos(m\pi t/L)$ and $\sin(m\pi t/L)$ are orthogonal to both $\cos(n\pi t/L)$ and $\sin(n\pi t/L)$ for all $m \neq m$; as a result, the best approximation of h in P_1 is zero

$$FS_1(h) = 0.$$

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4 Fourier transform

Here are some formulae that you may want to use:

$$\mathcal{F}(f)(\omega) \stackrel{\text{def}}{=} \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x)e^{i\omega x}dx, \qquad \mathcal{F}^{-1}(f)(x) = \int_{-\infty}^{+\infty} f(\omega)e^{-i\omega x}d\omega, \tag{3}$$

$$\mathcal{F}(f*g) = 2\pi\mathcal{F}(f)\mathcal{F}(g), \tag{4}$$

$$\mathcal{F}(e^{-\alpha|x|}) = \frac{1}{\pi}\frac{\alpha}{\omega^2+\alpha^2}, \qquad \mathcal{F}(\frac{2\alpha}{x^2+\alpha^2})(\omega) = e^{-\alpha|\omega|}, \tag{5}$$

$$\mathcal{F}(f(x-\beta))(\omega) = e^{i\beta\omega}\mathcal{F}(f)(\omega), \tag{6}$$

$$\mathcal{F}(e^{-\alpha x^2}) = \frac{1}{\sqrt{4\pi\alpha}}e^{-\frac{\omega^2}{4\alpha}} \tag{7}$$

$$\mathcal{F}(H(x)e^{-ax})(\omega) = \frac{1}{2\pi}\frac{1}{a-i\omega}, \qquad H \text{ is the Heaviside function} \tag{8}$$

$$\mathcal{F}(\mathrm{pv}(\frac{1}{x}))(\omega) = \frac{i}{2}\mathrm{sign}(\omega) = \frac{i}{2}(H(\omega)-H(-\omega)) \tag{9}$$

$$\mathcal{F}(\mathrm{sech}(ax)) = \frac{1}{2a}\mathrm{sech}(\frac{\pi}{2a}\omega), \qquad \mathrm{sech}(ax) \stackrel{\text{def}}{=} \frac{1}{\mathrm{ch}(x)} = \frac{2}{e^x+e^{-x}} \tag{10}$$

$$\cos(a)-\cos(b) = -2\sin(\tfrac{1}{2}(a+b))\sin(\tfrac{1}{2}(a-b)) \tag{11}$$

Question 41: (i) Let f be an integrable function on $(-\infty,+\infty)$. Prove that for all $a,b\in\mathbb{R}$, and for all $\xi\in\mathbb{R}$, $\mathcal{F}([e^{ibx}f(ax)])(\xi) = \frac{1}{a}\mathcal{F}(f)(\frac{\xi+b}{a})$.

Solution: The definition of the Fourier transform together with the change of variable $ax \longmapsto x'$ implies

$$\begin{aligned}\mathcal{F}[e^{ibx}f(ax)])(\xi) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(ax)e^{ibx}e^{i\xi x}dx \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(ax)e^{i(b+\xi)x}dx \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{1}{a}f(x')e^{i\frac{(\xi+b)}{a}x'}dx' \\ &= \tfrac{1}{a}\mathcal{F}(f)(\tfrac{\xi+b}{a}).\end{aligned}$$

(ii) Let c be a positive real number. Compute the Fourier transform of $f(x) = e^{-cx^2}\sin(bx)$.

Solution: Using the fact that $\sin(bx) = -i\frac{1}{2}(e^{ibx}-e^{-ibx})$, and setting $a=\sqrt{c}$ we infer that

$$\begin{aligned}f(x) &= \tfrac{1}{2i}e^{-(ax)^2}(e^{ibx}-e^{-ibx}) \\ &= \tfrac{1}{2i}e^{-(ax)^2}e^{ibx} - \tfrac{1}{2i}e^{-(ax)^2}e^{i(-b)x})\end{aligned}$$

and using (i) and (7) we deduce

$$\hat{f}(\xi) = \tfrac{1}{2i}\tfrac{1}{a}\tfrac{1}{\sqrt{4\pi}}(e^{-(\frac{\xi+b}{a})^2} - e^{-(\frac{\xi-b}{a})^2}).$$

In conclusion

$$\hat{f}(\xi) = \tfrac{1}{2i}\tfrac{1}{\sqrt{4\pi c}}(e^{-\frac{1}{c}(\xi+b)^2} - e^{-\frac{1}{c}(\xi-b)^2}).$$

Question 42: (a) Compute the Fourier transform of the function $f(x)$ defined by

$$f(x) = \begin{cases} 1 & \text{if } |x|\le 1 \\ 0 & \text{otherwise}\end{cases}$$

Solution: By definition

$$\mathcal{F}(f)(\omega) = \frac{1}{2\pi}\int_{-1}^{1} e^{i\xi\omega} == \frac{1}{2\pi}\frac{1}{i\omega}(e^{i\omega} - e^{-i\omega})$$
$$= \frac{1}{2\pi}\frac{2\sin(\omega)}{\omega}.$$

Hence

$$\mathcal{F}(f)(\omega) = \frac{1}{\pi}\frac{\sin(\omega)}{\omega}.$$

(b) Find the inverse Fourier transform of $g(\omega) = \frac{\sin(\omega)}{\omega}$.

Solution: Using (a) we deduce that $g(\omega) = \pi\mathcal{F}(f)(\omega)$, that is to say, $\mathcal{F}^{-1}(g)(x) = \pi\mathcal{F}^{-1}(\mathcal{F}(f))(x)$. Now, using the inverse Fourier transform, we deduce that $\mathcal{F}^{-1}(g)(x) = \pi f(x)$ at every point x where $f(x)$ is of class C^1 and $\mathcal{F}^{-1}(g)(x) = \frac{\pi}{2}(f(x^-) + f(x^+))$ at discontinuity points of f. As a result:

$$\mathcal{F}^{-1}(g)(x) = \begin{cases} \pi & \text{if } |x| < 1 \\ \frac{\pi}{2} & \text{at } |x| = 1 \\ 0 & \text{otherwise} \end{cases}$$

Question 43: Use the Fourier transform technique to solve the following PDE:

$$\partial_t u(x,t) + c\partial_x u(x,t) + \gamma u(x,t) = 0,$$

for all $x \in (-\infty, +\infty)$, $t > 0$, with $u(x,0) = u_0(x)$ for all $x \in (-\infty, +\infty)$.

Solution: By taking the Fourier transform of the PDE, one obtains

$$\partial_t \mathcal{F}(u) - i\omega c\mathcal{F}(y) + \gamma\mathcal{F}(y) = 0.$$

The solution is

$$\mathcal{F}(u)(\omega, t) = c(\omega)e^{i\omega ct - \gamma t}.$$

The initial condition implies that $c(\omega) = \mathcal{F}(u_0)(\omega)$:

$$\mathcal{F}(u)(\omega, t) = \mathcal{F}(u_0)(\omega)e^{i\omega ct}e^{-\gamma t}.$$

The shift lemma in turn implies that

$$\mathcal{F}(u)(\omega, t) = \mathcal{F}(u_0(x - ct))(\omega)e^{-\gamma t} = \mathcal{F}(u_0(x - ct)e^{-\gamma t})(\omega).$$

Applying the inverse Fourier transform gives:

$$u(x,t) = u_0(x - ct)e^{-\gamma t}.$$

Question 44: Solve by the Fourier transform technique the following equation: $\partial_{xx}\phi(x) - 2\partial_x\phi(x) + \phi(x) = H(x)e^{-x}$, $x \in (-\infty, +\infty)$, where $H(x)$ is the Heaviside function. (Hint: use the factorization $i\omega^3 + \omega^2 + i\omega + 1 = (1 + \omega^2)(1 + i\omega)$ and recall that $\mathcal{F}(f(x))(-\omega) = \mathcal{F}(f(-x))(\omega)$).

Solution: Applying the Fourier transform with respect to x gives

$$(-\omega^2 + 2i\omega + 1)\mathcal{F}(\phi)(\omega) = \mathcal{F}(H(x)e^{-x})(\omega) = \frac{1}{2\pi}\frac{1}{1 - i\omega}.$$

where we used (8). Then, using the hint gives

$$\mathcal{F}(\phi)(\omega) = \frac{1}{2\pi}\frac{1}{(1 - i\omega)(-\omega^2 + 2i\omega + 1)} = \frac{1}{2\pi}\frac{1}{i\omega^3 + \omega^2 + i\omega + 1}$$
$$= \frac{1}{2\pi}\frac{1}{1 + \omega^2}\frac{1}{1 + i\omega}.$$

We now use again (8) and (5) to obtain

$$\mathcal{F}(\phi)(\omega) = \pi\frac{1}{\pi}\frac{1}{1+\omega^2}\frac{1}{2\pi}\frac{1}{1-i(-\omega)} = \pi\mathcal{F}(e^{-|x|})(\omega)\mathcal{F}(H(x)e^{-x})(-\omega).$$

Now we use $\mathcal{F}(H(x)e^{-x})(-\omega) = \mathcal{F}(H(-x)e^{x})(\omega)$ and we finally have

$$\mathcal{F}(\phi)(\omega) = \pi\mathcal{F}(e^{-|x|})(\omega)\mathcal{F}(H(-x)e^{x})(\omega).$$

The Convolution Theorem (4) gives

$$\mathcal{F}(\phi)(\omega) = \pi\frac{1}{2\pi}\mathcal{F}(e^{-|x|} * (H(-x)e^{x}))(\omega).$$

We obtain ϕ by using the inverse Fourier transform

$$\phi(x) = \frac{1}{2}e^{-|x|} * (H(-x)e^{x}),$$

i.e.,

$$\phi(x) = \int_{-\infty}^{+\infty} \frac{1}{2}e^{-|x-y|}H(-y)e^{y}\mathsf{d}y$$

and recalling that H is the Heaviside function we finally have

$$\phi(x) = \frac{1}{2}\int_{-\infty}^{0} e^{y-|x-y|}\mathsf{d}y = \begin{cases} \frac{1}{4}e^{-x} & \text{if } x \geq 0 \\ (\frac{1}{4}-x)e^{x} & \text{if } x \leq 0. \end{cases}$$

Question 45: Use the Fourier transform technique to solve the following ODE $y''(x) - y(x) = f(x)$ for $x \in (-\infty, +\infty)$, with $y(\pm\infty) = 0$, where f is a function such that $|f|$ is integrable over $\mathbb{R}$.

Solution: By taking the Fourier transform of the ODE, one obtains

$$-\omega^2\mathcal{F}(y) - \mathcal{F}(y) = \mathcal{F}(f).$$

That is

$$\mathcal{F}(y) = -\mathcal{F}(f)\frac{1}{1+\omega^2}.$$

and the convolution Theorem, see (4), together with (5) gives

$$\mathcal{F}(y) = -\pi\mathcal{F}(f)\mathcal{F}(e^{-|x|}) = -\frac{1}{2}\mathcal{F}(f * e^{-|x|}).$$

Applying $\mathcal{F}^{-1}$ on both sides we obtain

$$y(x) = -\frac{1}{2}f * e^{-|x|} = -\frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-z|}f(z)dz$$

That is

$$y(x) = -\frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-z|}f(z)dz.$$

Question 46: Use the Fourier transform method to compute the solution of $u_{tt} - a^2u_{xx} = 0$, where $x \in \mathbb{R}$ and $t \in (0, +\infty)$, with $u(0,x) = f(x) := \sin^2(x)$ and $u_t(0,x) = 0$ for all $x \in \mathbb{R}$.

Solution: Take the Fourier transform in the x direction:

$$\mathcal{F}(u)_{tt} + \omega^2a^2\mathcal{F}(u) = 0.$$

This is an ODE. The solution is

$$\mathcal{F}(u)(t,\omega) = c_1(\omega)\cos(\omega at) + c_2(\omega)\sin(\omega at).$$

The initial boundary conditions give

$$\mathcal{F}(u)(0,\xi) = \mathcal{F}(f)(\omega) = c_1(\omega)$$

and $c_2(\omega) = 0$. Hence

$$\mathcal{F}(t,\omega) = \mathcal{F}(f)(\omega)\cos(\omega at) = \frac{1}{2}\mathcal{F}(f)(\omega)(e^{ia\omega t} + e^{-ia\omega t}).$$

Using the shift lemma, we infer that

$$\mathcal{F}(t,\omega) = \frac{1}{2}(\mathcal{F}(f(x-at))(\omega) + \mathcal{F}(f(x+at))(\omega)).$$

Usin the inverse Fourier transform, we finaly conclude that

$$u(t,x) = \frac{1}{2}(f(x-at) + f(x+at)) = \frac{1}{2}(\sin^2(x+at) + \sin^2(x-at)).$$

Note that this is the D'Alembert formula.

Question 47: Use the Fourier transform method to compute the solution of $u_{tt} - a^2 u_{xx} = 0$, where $x \in \mathbb{R}$ and $t \in (0,+\infty)$, with $u(0,x) = f(x) := \cos^2(x)$ and $u_t(0,x) = 0$ for all $x \in \mathbb{R}$.

Solution: Take the Fourier transform in the x direction:

$$\mathcal{F}(u)_{tt} + \omega^2 a^2 \mathcal{F}(u) = 0.$$

This is an ODE. The solution is

$$\mathcal{F}(u)(t,\omega) = c_1(\omega)\cos(\omega at) + c_2(\omega)\sin(\omega at).$$

The initial boundary conditions give

$$\mathcal{F}(u)(0,\xi) = \mathcal{F}(f)(\omega) = c_1(\omega)$$

and $c_2(\omega) = 0$. Hence

$$\mathcal{F}(t,\omega) = \mathcal{F}(f)(\omega)\cos(\omega at) = \frac{1}{2}\mathcal{F}(f)(\omega)(e^{ia\omega t} + e^{-ia\omega t}).$$

Using the shift lemma (i.e., formula (6)) we obtain

$$u(t,x) = \frac{1}{2}(f(x-at) + f(x+at)) = \frac{1}{2}(\cos^2(x+at) + \cos^2(x-at)).$$

Note that this is the D'Alembert formula.

Question 48: Solve the integral equation: $f(x) + \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{f(y)}{(x-y)^2+1}dy = \frac{1}{x^2+4} + \frac{1}{x^2+1}$, for all $x \in (-\infty,+\infty)$.

Solution: The equation can be re-written

$$f(x) + \frac{1}{2\pi} f * \frac{1}{x^2+1} = \frac{1}{x^2+4} + \frac{1}{x^2+1}.$$

We take the Fourier transform of the equation and we apply the Convolution Theorem (see (4))

$$\mathcal{F}(f) + \frac{1}{2\pi}2\pi\mathcal{F}(\frac{1}{x^2+1})\mathcal{F}(f) = \mathcal{F}(\frac{1}{x^2+4}) + \mathcal{F}(\frac{1}{x^2+1}).$$

Using (5), we obtain

$$\mathcal{F}(f) + \frac{1}{2}e^{-|\omega|}\mathcal{F}(f) = \frac{1}{4}e^{-2|\omega|} + \frac{1}{2}e^{-|\omega|},$$

which gives

$$\mathcal{F}(f)(1 + \frac{1}{2}e^{-|\omega|}) = \frac{1}{2}e^{-|\omega|}(\frac{1}{2}e^{-|\omega|} + 1).$$

We then deduce

$$\mathcal{F}(f) = \frac{1}{2}e^{-|\omega|}.$$

Taking the inverse Fourier transform, we finally obtain $f(x) = \frac{1}{x^2+1}$.

Question 49: Solve the following integral equation (Hint: solution is short):

$$\int_{-\infty}^{+\infty} f(y)f(x-y)\mathrm{d}y - 2\sqrt{2}\int_{-\infty}^{+\infty} e^{-\frac{y^2}{2\pi}} f(x-y)\mathrm{d}y = -2\pi e^{-\frac{x^2}{4\pi}}. \qquad \forall x \in \mathbb{R}.$$

Solution: This equation can be re-written using the convolution operator:

$$f * f - 2\sqrt{2}e^{-\frac{x^2}{2\pi}} * f = -2\pi e^{-\frac{x^2}{4\pi}}.$$

We take the Fourier transform and use the convolution theorem (4) together with (7) to obtain

$$\begin{aligned} 2\pi\mathcal{F}(f)^2 - 2\pi 2\sqrt{2}\mathcal{F}(f)\frac{1}{\sqrt{4\pi\frac{1}{2\pi}}}e^{-\omega^2\frac{1}{4\frac{1}{2\pi}}} &= -2\pi\frac{1}{\sqrt{4\pi\frac{1}{4\pi}}}e^{-\omega^2\frac{1}{4\frac{1}{4\pi}}} \\ \mathcal{F}(f)^2 - 2\mathcal{F}(f)e^{-\omega^2\frac{\pi}{2}} + e^{-\omega^2\pi} &= 0 \\ (\mathcal{F}(f) - e^{-\omega^2\frac{\pi}{2}})^2 &= 0. \end{aligned}$$

This implies

$$\mathcal{F}(f) = e^{-\omega^2\frac{\pi}{2}}.$$

Taking the inverse Fourier transform, we obtain

$$f(x) = \sqrt{2}e^{-\frac{x^2}{2\pi}}.$$

Question 50: Solve the following integral equation (Hint: $x^2 - 3xa + 2a^2 = (x-a)(x-2a)$):

$$\int_{-\infty}^{+\infty} f(y)f(x-y)\mathrm{d}y - 3\sqrt{2}\int_{-\infty}^{+\infty} e^{-\frac{y^2}{2\pi}} f(x-y)\mathrm{d}y = -4\pi e^{-\frac{x^2}{4\pi}}. \qquad \forall x \in \mathbb{R}.$$

Solution: This equation can be re-written using the convolution operator:

$$f * f - 3\sqrt{2}e^{-\frac{x^2}{2\pi}} * f = -4\pi e^{-\frac{x^2}{4\pi}}.$$

We take the Fourier transform and use (7) to obtain

$$\begin{aligned} 2\pi\mathcal{F}(f)^2 - 2\pi 3\sqrt{2}\mathcal{F}(f)\frac{1}{\sqrt{4\pi\frac{1}{2\pi}}}e^{-\omega^2\frac{1}{4\frac{1}{2\pi}}} &= -4\pi\frac{1}{\sqrt{4\pi\frac{1}{4\pi}}}e^{-\omega^2\frac{1}{4\frac{1}{4\pi}}} \\ \mathcal{F}(f)^2 - 3\mathcal{F}(f)e^{-\omega^2\frac{\pi}{2}} + 2e^{-\omega^2\pi} &= 0 \\ (\mathcal{F}(f) - e^{-\omega^2\frac{\pi}{2}})(\mathcal{F}(f) - 2e^{-\omega^2\frac{\pi}{2}}) &= 0. \end{aligned}$$

This implies

$$\text{either} \quad \mathcal{F}(f) = e^{-\omega^2\frac{\pi}{2}}, \quad \text{or} \quad \mathcal{F}(f) = 2e^{-\omega^2\frac{\pi}{2}}.$$

Taking the inverse Fourier transform, we obtain

$$\text{either} \quad f(x) = \sqrt{2}e^{-\frac{x^2}{2\pi}}, \quad \text{or} \quad f(x) = 2\sqrt{2}e^{-\frac{x^2}{2\pi}}.$$

Question 51: Solve the integral equation: $f(x) + \frac{3}{2}\int_{-\infty}^{+\infty} e^{-|x-y|}f(y)dy = e^{-|x|}$, for all $x \in (-\infty, +\infty)$.

Solution: The equation can be re-written

$$f(x) + \frac{3}{2}e^{-|x|} * f = e^{-|x|}.$$

We take the Fourier transform of the equation and apply the Convolution Theorem (see (4))

$$\mathcal{F}(f) + \frac{3}{2}2\pi\mathcal{F}(e^{-|x|})\mathcal{F}(f) = \mathcal{F}(e^{-|x|}).$$

Using (5), we obtain

$$\mathcal{F}(f) + 3\pi\frac{1}{\pi}\frac{1}{1+\omega^2}\mathcal{F}(f) = \frac{1}{\pi}\frac{1}{1+\omega^2},$$

which gives

$$\mathcal{F}(f)\frac{\omega^2+4}{1+\omega^2} = \frac{1}{\pi}\frac{1}{1+\omega^2}.$$

We then deduce

$$\mathcal{F}(f) = \frac{1}{\pi}\frac{1}{4+\omega^2} = \frac{1}{2}\mathcal{F}(e^{-2|x|}).$$

Taking the inverse Fourier transform, we finally obtain $f(x) = \frac{1}{2}e^{-2|x|}$.

Question 52: Solve the following integral equation $\int_{-\infty}^{+\infty} e^{-(x-y)^2}g(y)dy = e^{-\frac{1}{2}x^2}$ for all $x \in (-\infty,+\infty)$, i.e., find the function g that solves the above equation.

Solution: The left-hand side of the equation is a convolution; hence,

$$e^{-x^2} * g(x) = e^{-\frac{1}{2}x^2}.$$

By taking the Fourier transform, we obtain

$$2\pi\frac{1}{\sqrt{4\pi}}e^{-\frac{1}{4}\omega^2}\mathcal{F}g(\omega) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\omega^2}.$$

That gives

$$\mathcal{F}(g)(\omega) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{4}\omega^2}.$$

By taking the inverse Fourier transform, we deduce

$$g(x) = \frac{\sqrt{4\pi}}{\sqrt{2\pi}}e^{-x^2} = \sqrt{\frac{2}{\pi}}e^{-x^2}.$$

Question 53: Solve the integral equation: $\int_{-\infty}^{+\infty} f(y)f(x-y)dy = \frac{4}{x^2+4}$, for all $x \in (-\infty,+\infty)$. How many solutions did you find?

Solution: The equation can be re-written

$$f * f = \frac{4}{x^2+4}.$$

We take the Fourier transform of the equation and apply the Convolution Theorem (see (4))

$$2\pi\mathcal{F}(f)^2 = \mathcal{F}(\frac{4}{x^2+4})$$

Using (5), we obtain

$$2\pi\mathcal{F}(f)^2 = e^{-2|\omega|}.$$

which gives

$$\mathcal{F}(f) = \pm\frac{1}{\sqrt{2\pi}}e^{-|\omega|}.$$

Taking the inverse Fourier transform, we finally obtain

$$f(x) = \pm \frac{1}{\sqrt{2\pi}} \frac{2}{x^2+1}.$$

We found two solutions: a positive one and a negative one.

Question 54: Use the Fourier transform method to solve the equation $\partial_t u + \frac{2t}{1+t^2}\partial_x u = 0$, $u(x,0) = u_0(x)$, in the domain $x \in (-\infty,+\infty)$ and $t > 0$.

Solution: We take the Fourier transform of the equation with respect to x

$$\begin{aligned} 0 &= \partial_t \mathcal{F}(u) + \mathcal{F}(\frac{2t}{1+t^2}\partial_x u) \\ &= \partial_t \mathcal{F}(u) + \frac{2t}{1+t^2}\mathcal{F}(\partial_x u) \\ &= \partial_t \mathcal{F}(u) - i\omega\frac{2t}{1+t^2}\mathcal{F}(u). \end{aligned}$$

This is a first-order linear ODE:

$$\frac{\partial_t \mathcal{F}(u)}{\mathcal{F}(u)} = i\omega\frac{2t}{1+t^2} = i\omega\frac{\mathsf{d}}{\mathsf{d}t}(\log(1+t^2))$$

The solution is

$$\mathcal{F}(u)(\omega,t) = K(\omega)e^{i\omega\log(1+t^2)}.$$

Using the initial condition, we obtain

$$\mathcal{F}(u_0)(\omega) = \mathcal{F}(u)(\omega,0) = K(\omega).$$

The shift lemma (see formula (6)) implies

$$\mathcal{F}(u)(\omega,t) = \mathcal{F}(u_0)(\omega)e^{i\omega\log(1+t^2)} = \mathcal{F}(u_0(x-\log(1+t^2))),$$

Applying the inverse Fourier transform finally gives

$$u(x,t) = u_0(x-\log(1+t^2)).$$

Question 55: Solve the integral equation:

$$\int_{-\infty}^{+\infty}\left(f(y) - \sqrt{2}e^{-\frac{y^2}{2\pi}} - \frac{1}{1+y^2}\right)f(x-y)dy = -\int_{-\infty}^{+\infty}\frac{\sqrt{2}}{1+y^2}e^{-\frac{(x-y)^2}{2\pi}}\mathrm{d}y, \qquad \forall x \in (-\infty,+\infty).$$

(Hint: there is an easy factorization after applying the Fourier transform.)

Solution: The equation can be re-written

$$f * (f - \sqrt{2}e^{-\frac{x^2}{2\pi}} - \frac{1}{1+x^2}) = -\frac{1}{1+x^2} * \sqrt{2}e^{-\frac{x^2}{2\pi}}.$$

We take the Fourier transform of the equation and apply the Convolution Theorem (see (4))

$$2\pi\mathcal{F}(f)\left(\mathcal{F}(f) - \sqrt{2}\mathcal{F}(e^{-\frac{x^2}{2\pi}}) - \mathcal{F}(\frac{1}{1+x^2})\right) = -2\pi\mathcal{F}(\frac{1}{1+x^2})\sqrt{2}\mathcal{F}(e^{-\frac{x^2}{2\pi}})$$

Using (5), (7) we obtain

$$\begin{aligned} \sqrt{2}\mathcal{F}(e^{-\frac{x^2}{2\pi}}) &= \sqrt{2}\frac{1}{\sqrt{4\pi\frac{1}{2\pi}}}e^{-\frac{\omega^2}{4\frac{1}{2\pi}}} = e^{-\frac{\pi\omega^2}{2}} \\ \mathcal{F}(\frac{1}{1+x^2}) &= \frac{1}{2}e^{-|\omega|}, \end{aligned}$$

which gives
$$\mathcal{F}(f)\left(\mathcal{F}(f) - e^{-\frac{\pi\omega^2}{2}} - \frac{1}{2}e^{-|\omega|}\right) = -\frac{1}{2}e^{-|\omega|}e^{-\frac{\pi\omega^2}{2}}.$$
This equation can also be re-written as follows
$$\mathcal{F}(f)^2 - \mathcal{F}(f)e^{-\frac{\pi\omega^2}{2}} - \mathcal{F}(f)\frac{1}{2}e^{-|\omega|} + \frac{1}{2}e^{-|\omega|}e^{-\frac{\pi\omega^2}{2}} = 0,$$
and can be factorized as follows:
$$(\mathcal{F}(f) - e^{-\frac{\pi\omega^2}{2}})(\mathcal{F}(f) - \frac{1}{2}e^{-|\omega|}) = 0.$$
This means that either $\mathcal{F}(f) = e^{-\frac{\pi\omega^2}{2}}$ or $\mathcal{F}(f) = \frac{1}{2}e^{-|\omega|}$. Taking the inverse Fourier transform, we finally obtain two solutions
$$f(x) = \sqrt{2}e^{-\frac{x^2}{2\pi}}, \quad \text{or} \quad f(x) = \frac{1}{1+x^2}.$$
Another solution consists of observing that the equation can also be re-written
$$\mathcal{F}(f)^2 - \sqrt{2}\mathcal{F}(e^{-\frac{x^2}{2\pi}})\mathcal{F}(f) - \mathcal{F}(\frac{1}{1+x^2})\mathcal{F}(f) + \mathcal{F}(\frac{1}{1+x^2})\sqrt{2}\mathcal{F}(e^{-\frac{x^2}{2\pi}}) = 0$$

5 Wave equation

Question 56: Consider the following wave equation

$$\partial_{tt}w - c^2\partial_{xx}w = 0, \quad x > 0,\ t > 0$$
$$w(x,0) = f(x), \quad x > 0, \qquad \partial_t w(x,0) = 0, \quad x > 0, \quad \text{and} \quad w(0,t) = 0, \quad t > 0.$$

(a) Solve the equation (Hint: recall that the solution can always be put in the form $F(x-ct)+G(x+ct)$)

Solution: If $x - ct > 0$, we can apply D'Alembert's formula $u(x,t) = F(x-ct) + G(x+ct)$

$$F(z) = \frac{1}{2}f(z) + \frac{1}{2c}\int_z^0 g(\tau)d\tau, \quad G(z) = \frac{1}{2}f(z) + \frac{1}{2c}\int_0^z g(\tau)d\tau,$$

where $g(x) = \partial_t w(x,0) = 0$. In other words

$$w(x,t) = \frac{1}{2}(f(x-ct) + f(x+ct)), \quad \text{if } x \geq ct.$$

If $x - ct < 0$ we apply the boundary condition at $x = 0$

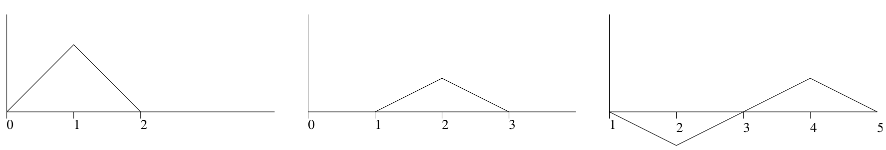
$$w(0,t) = 0 = F(-ct) + G(ct), \quad \forall t \geq 0.$$

This means $f(-z) = -G(z) = -\frac{1}{2}f(z)$ for all $z \geq 0$. In other words we have obtained

$$w(x,t) = \frac{1}{2}(-f(-x+ct) + f(x+ct)), \quad \text{if } x \leq ct.$$

(b) We now set $c = 1$, $f(x) = x$, if $x \in [0,1]$, $f(x) = 2 - x$, if $x \in [1,2]$, and $f(x) = 0$ otherwise. Draw the graph of the solution at $t = 0$, $t = 1$, and $t = 2$ (draw three different graphs).

Solution:



Question 57: Let u solve the wave equation $\partial_{tt}u - c^2\partial_{xx}u = 0$, $x \in [0,L]$, $t \geq 0$, with $u(x,0) = f(x)$, $u_t(x,0) = 0$. Let $F(x)$, $x \in (-\infty,+\infty)$ be the odd periodic extension of f. Prove that $u(x,t) = \frac{1}{2}(F(x+ct)+F(x-ct))$ is the solution (do not spend to much time proving $u(L,t) = 0$ if you do not remember how to do it).

Solution: Clearly $u(x,t)$ solve the PDE $\partial_{tt}u - c^2\partial_{xx}u = 0$, $x \in [0,L]$, $t \geq 0$. Moreover $u(x,0) = \frac{1}{2}(F(x)+F(x)) = f(x)$, for all $x \in [0,L]$. Moreover $\partial_t u(x,0) = \frac{1}{2}(cF(x) - cF(x)) = 0$, for all $x \in [0,L]$. Moreover $u(0,t) = \frac{1}{2}(F(ct) + F(-ct)) = 0$ since F is odd. Let us compute $u(L,t) = \frac{1}{2}(F(L+ct) + F(L-ct))$. Observe $-(L+ct) = -L - ct = L - ct - 2L$, that is to say $F(-(L+ct)) = F(L-ct-2L) = F(L-ct)$, since F is $2L$-periodic. But since F is odd, we have $-F(L+ct) = F(-(L+ct)) = F(L-ct)$, i.e., $u(L,t) = 0$.

Question 58: Let u be a solution of the wave equation $\partial_{tt}u - c^2\partial_{xx}u = 0$, $x \in [0,L]$, $t \geq 0$. Let $E = \frac{1}{2}\int_0^L (\partial_t u)^2 dx + \frac{c^2}{2}\int_0^L (\partial_x u)^2 dx$.
(a) Compute the time derivative of E.

Solution: Multiply the equation by $\partial_t u$ and integrate over the domain. It is clear that

$$\frac{dE}{dt} = \partial_x u(x,t)\partial_t u(x,t)|_0^L.$$

(b) In addition to being a solution of the wave equation, assume that $u(0,t)=0$, $\partial_x u(L,t)=0$, and $u(x,0)=f(x)$, $\partial_t u(x,0)=g(x)$, where f and g are two given functions. Prove that if a solution to this problem exists, then it is unique.

Solution: Let u_1, u_2 be two solutions and let $\phi = u_1 - u_2$. Then ϕ solves the homogeneous problem. From (a) we infer $E(t)=E(0)=0$ where E is the energy for ϕ. Hence $\partial_x\phi = 0$, which means $\phi(x,t)=\psi(t)$. But $\phi(0,t)=\psi(t)=0$. In conclusion $\phi=0$, i.e., $u_1=u_2$.

Question 59: Solve the PDE

$$\begin{aligned} &u_{tt} - a^2 u_{xx} = 0, && -\infty < x < +\infty, \quad 0 \le t,\\ &u(x,0) = \sin(x), \quad u_t(x,0) = a\cos(x), && -\infty < x < +\infty. \end{aligned}$$

Solution: Apply D'Alembert's Formula. $u(x,t) = \sin(x+at)$.

Question 60: Solve the PDE

$$\begin{aligned} &u_{tt} - a^2 u_{xx} = 0, && -\infty < x < +\infty, \quad 0 \le t,\\ &u(0,t) = 0, && 0 \le t,\\ &u(x,0) = x^3, \quad u_t(x,0) = 0, && -\infty < x < +\infty. \end{aligned}$$

Solution: Use the odd extension of x^3, that is, x^3. Than, apply D'Alembert's Formula. $u(x,t) = \frac{1}{2}[(x+at)^3 + (x-at)^3]$.

Question 61: Solve the PDE

$$\begin{aligned} &u_{tt} - a^2 u_{xx} = 0, && -\infty < x < +\infty, \quad 0 \le t,\\ &u(x,0) = \cos(x), \quad u_t(x,0) = -a\sin(x), && -\infty < x < +\infty. \end{aligned}$$

Solution: Apply D'Alembert's Formula.

$$\begin{aligned} u(x,t) &= \tfrac{1}{2}(\cos(x-at) + \cos(x+at)) - \tfrac{1}{2a}\int_{x-at}^{x+at} a\sin(\xi)d\xi \\ &= \tfrac{1}{2}(\cos(x-at) + \cos(x+at)) + \tfrac{1}{2}(\cos(x+at) - \cos(x-at)) \\ &= \cos(x+at). \end{aligned}$$

Hence $u(x,t) = \cos(x+at)$.

Question 62: Solve the PDE

$$\begin{aligned} &u_{tt} - u_{xx} = 0, && 0 < x < +\infty, \quad 0 < t,\\ &u(0,t) = 0, \quad u(1,t) = 0 && 0 < t,\\ &u(x,0) = \sin(\pi x), \quad u_t(x,0) = 0, && 0 < x < +\infty. \end{aligned}$$

Solution: We have to define the odd extension of $\sin(\pi x)$ on $(-1,+1)$. Clearly $\sin(\pi x)$ is the odd extension. Now we define the periodic extension of $\sin(\pi x)$ over the entire real line. Clearly $\sin(\pi x)$ is the extension in question. The D'Alembert formula, which is valid on the entire real line, gives

$$\begin{aligned} u(x,t) &= \tfrac{1}{2}(\sin(\pi(x-t)) + \sin(\pi(x+t)) \\ &= \tfrac{1}{2}((\cos(\pi t)\sin(\pi x) - \sin(\pi t)\cos(\pi x)) + \tfrac{1}{2}((\cos(\pi t)\sin(\pi x) + \sin(\pi t)\cos(\pi x)) \\ &= \cos(\pi t)\sin(\pi x). \end{aligned}$$

Hence $u(x,t) = \cos(\pi t)\sin(\pi x)$ for all $x \in (0,1)$, $t > 0$.

Question 63: Consider the PDE

$$\begin{aligned} &u_{tt} - u_{xx} = 0, && 0 \le x \le 2, \quad 0 < t,\\ &u(0,t) = 0, \quad u(2,t) = 0 && 0 < t,\\ &u_t(x,0) = 0, \quad u(x,0) = f(x) := \begin{cases} x & 0 \le x \le 1, \\ 2-x & 1 \le x \le 2. \end{cases} && 0 < x < +2. \end{aligned}$$

(a) Give u(x,t) for all $x \in [0,+2]$, $t > 0$. (Hint use an extension technique).

Solution: We notice first that the wave speed is 1. We define f_o to be the odd extension of f over $(-2,+2)$, the we define f_{op} to be the periodic extension of f_o over $(-\infty,+\infty)$ with period 4. From class we know that the solution to the above problem is given by the D'Alembert formula

$$u(x,t) = \frac{1}{2}(f_{op}(x-t) + f_{op}(x+t)).$$

(b) Using (a), compute $u(x,\frac{1}{2})$, for all $x \in [0,+2]$.

Solution: We have to compute $f_{op}(x-\frac{1}{2})$ and $f_{op}(x+\frac{1}{2})$.

Case 1: $0 \le x \le \frac{1}{2}$. Then $-\frac{1}{2} \le x - \frac{1}{2} \le 0$ and by definition of f_{op}, $f_{op}(x-\frac{1}{2}) = -f(-x+\frac{1}{2}) = x - \frac{1}{2}$. We also have $\frac{1}{2} \le x + \frac{1}{2} \le 1$, which means $f_{op}(x+\frac{1}{2}) = f(x+\frac{1}{2}) = x + \frac{1}{2}$. Finally $u(x,\frac{1}{2}) = \frac{1}{2}(x - \frac{1}{2} + x + \frac{1}{2}) = x$ for all $x \in [0,\frac{1}{2}]$.

Case 2: $\frac{1}{2} \le x \le \frac{3}{2}$. Then $0 \le x - \frac{1}{2} \le 1$ and $f_{op}(x-\frac{1}{2}) = f(x-\frac{1}{2}) = x - \frac{1}{2}$. We also have $1 \le x + \frac{1}{2} \le 2$, which means $f_{op}(x+\frac{1}{2}) = f(x+\frac{1}{2}) = 2 - (x+\frac{1}{2}) = -x + \frac{3}{2}$. Finally $u(x,\frac{1}{2}) = \frac{1}{2}(x - \frac{1}{2} - x + \frac{3}{2}) = \frac{1}{2}$ for all $x \in [\frac{1}{2},\frac{3}{2}]$.

Case 3: $\frac{3}{2} \le x \le 2$. Then $1 \le x - \frac{1}{2} \le \frac{3}{2}$ and $f_{op}(x-\frac{1}{2}) = f(x-\frac{1}{2}) = 2 - (x-\frac{1}{2}) = \frac{5}{2} - x$. We also have $2 \le x+\frac{1}{2} \le \frac{5}{2}$, which means by periodicity that $f_{op}(x+\frac{1}{2}) = f_{op}(x+\frac{1}{2}-4) = f_{op}(x-\frac{7}{2})$. Now we observe that $-2 \le x - \frac{7}{2} \le -\frac{3}{2}$, which means $f_{op}(x+\frac{1}{2}) = f_{op}(x-\frac{7}{2}) = -f(\frac{7}{2}-x) = -(2-(\frac{7}{2}-x)) = -(-\frac{3}{2}+x) = \frac{3}{2} - x$. In conclusion $u(x,\frac{1}{2}) = \frac{1}{2}(\frac{5}{2} - x + \frac{3}{2} - x) = 2 - x$ for all $x \in [\frac{3}{2},2]$.

Conclusion: We now put everything together

$$u(x,\frac{1}{2}) = \begin{cases} x, & x \in [0,\frac{1}{2}], \\ \frac{1}{2}, & x \in [\frac{1}{2},\frac{3}{2}], \\ 2-x, & x \in [\frac{3}{2},2]. \end{cases} .$$

Question 64: Consider the equation $u''(x) = f(x)$ for $x \in (0,1)$ with $u(0) = 1$ and $u'(1) = 1$. Let $G(x,x_0)$ be the associated Green's function.
(a) Give an expression of $u(x)$ in terms of G, f and the boundary data.

Solution: The Green's function is defined by

$$G''(x,x_0) = \delta(x-x_0), \quad G(0,x_0) = 0, \quad G'(1,x_0) = 0.$$

We multiply the equation by u and we integrate (in the distribution sense),

$$\int_0^1 G''(x,x_0)u(x)dx = u(x_0).$$

We integrates by parts twice and we obtain,

$$\begin{aligned} u(x_0) &= -\int_0^1 G'(x,x_0)u'(x)dx + G'(1,x_0)u(1) - G'(0,x_0)u(0) \\ &= \int_0^1 G(x,x_0)u''(x)dx - G(1,x_0)u'(1) + G(0,x_0)u'(0) + G'(1,x_0)u(1) - G'(0,x_0)u(0). \end{aligned}$$

Then, using the boundary conditions for G and u, we obtain

$$u(x_0) = \int_0^1 G(x,x_0)f(x)dx - G'(0,x_0) - G(1,x_0), \quad \forall x_0 \in (0,1).$$

(b) Compute $G(x, x_0)$.

Solution: For $x < x_0$ we have

$$G(x, x_0) = ax + b.$$

The boundary condition $G(0, x_0) = 0$ implies $b = 0$. For $x_0 < x$ we have

$$G(x, x_0) = cx + d.$$

The boundary condition $G'(1, x_0) = 0$ implies $c = 0$. Moreover we have

$$1 = \lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} G''(x, x_0) dx = G'(x_0^+, x_0) - G'(x_0^-, x_0) = -a,$$

meaning $a = -1$. The continuity of G at x_0 implies

$$ax_0 = d,$$

implying $d = -x_0$. As a result,

$$G(x, x_0) = \begin{cases} -x, & \text{if } 0 \leq x \leq x_0, \\ -x_0, & \text{if } x_0 \leq x \leq 1. \end{cases}$$

Question 65: Solve the wave equation on the semi-infinite domain $(0, +\infty)$,

$$\partial_{tt} w - 4\partial_{xx} w = 0, \quad x \in (0, +\infty),\ t > 0$$
$$w(x, 0) = (1 + x^2)^{-1}, \quad x \in (0, +\infty); \qquad \partial_t w(x, 0) = 0, \quad x \in (0, +\infty); \quad \text{and} \quad \partial_x w(0, t) = 0, \quad t > 0.$$

(Hint: Consider a particular extension of w over $\mathbb{R}$)

Solution: We define $f(x) = (1 + x^2)^{-1}$ and its even extension $f_e(x)$ on $-\infty, +\infty$. Let w_e be the solution to the wave equation over the entire real line with f_e as initial data:

$$\partial_{tt} w_e - 4\partial_{xx} w_e = 0, \quad x \in \mathbb{R},\ t > 0$$
$$w_e(x, 0) = f_e(x), \quad x > 0, \qquad \partial_t w_e(x, 0) = 0, \quad x \in \mathbb{R}.$$

The solution to this problem is given by the D'Alembert formula

$$w_e(x, t) = \frac{1}{2}(f_e(x - 2t) + f_e(x + 2t)), \qquad \text{for all } x \in \mathbb{R} \text{ and } t \geq 0.$$

Let x be positive. Then $w(x, t) = w_e(x, t)$ for all $x \in (0, +\infty)$, since by construction $\partial_x w_e(0, t) = 0$ for all times.

Case 1: If $x - 2t > 0$, $f_e(x - 2t) = f(x - 2t)$; as a result

$$w(x, t) = \frac{1}{2}(f(x - 2t) + f(x + 2t), \quad \text{If } x - 2t > 0.$$

Case 2: If $x - 2t < 0$, $f_e(x - 2t) = f(-x + 2t)$; as a result

$$w(x, t) = \frac{1}{2}(f(-x + 2t) + f(x + 2t)), \quad \text{If } x - 2t < 0.$$

Note that actually $f_e(x) = (1 + x^2)^{-1}$; as a result, the solution can also be re-written as follows:

$$w(x, t) = \frac{1}{2}\left(\frac{1}{1 + (x - 2t)^2} + \frac{1}{1 + (x + 2t)^2}\right).$$

Question 66: Solve the PDE

$$\begin{aligned}
&u_{tt} - u_{xx} = 0, && 0 \le x \le 1, \quad 0 < t,\\
&\partial_x u(0,t) = 0, \quad \partial_x u(1,t) = 0 && 0 < t,\\
&u(x,0) = \cos(\pi x), \quad u_t(x,0) = 0, && 0 < x < +1.
\end{aligned}$$

(Hint: Consider the periodic extension over $\mathbb{R}$ of a particular extension of u over $[-1+1]$).

Solution: The even extension of u over $[-1+1]$, say u_e, satisfies the PDE and the initial conditions, and always satisfies $\partial_x u_e(0,t) = 0$, $\partial_x u_e(1,t) = 0$. Since $\partial u_e(1,t)$, we deduce $\partial u_e(-1,t) = 0$. This means that the periodic extension of u_e, says u_p, is smooth and also satisfies the PDE plus the initial conditions. By construction $\partial_x u_p(0,t) = 0$ and $\partial_x u_e(1,t) = 0$. As a result, we can obtain u by computing the solution of the wave equation on $\mathbb{R}$ using the periodic extension over $\mathbb{R}$ of the even extension of the initial data over $[-1+1]$, i.e., $u = u_p|_{[0,1]}$

We have to define the even extension of $\cos(\pi x)$ on $(-1,+1)$. Clearly $\cos(\pi x)$ is the even extension. Now we define the periodic extension of $\cos(\pi x)$ over the entire real line. Clearly $\cos(\pi x)$ is the extension in question. The D'Alembert formula, which is valid on the entire real line, gives

$$\begin{aligned}
u(x,t) &= \tfrac{1}{2}(\cos(\pi(x-t)) + \cos(\pi(x+t))\\
&= \tfrac{1}{2}((\cos(\pi t)\cos(\pi x) + \sin(\pi t)\sin(\pi x)) + \tfrac{1}{2}((\cos(\pi t)\cos(\pi x) - \sin(\pi t)\sin(\pi x))\\
&= \cos(\pi t)\cos(\pi x).
\end{aligned}$$

Hence $u(x,t) = \cos(\pi t)\cos(\pi x)$ for all $x \in (0,1)$, $t > 0$.

Question 67: Consider the following wave equation

$$\begin{aligned}
&\partial_{tt} w - 4\partial_{xx} w = 0, \quad x > 0,\ t > 0\\
&w(x,0) = x(1+x^2)^{-1}, \quad x > 0, \qquad \partial_t w(x,0) = 0, \quad x > 0, \quad \text{and} \quad w(0,t) = 0, \quad t > 0.
\end{aligned}$$

(a) Solve the equation.

Solution: We define $f(x) = x(1+x^2)^{-1}$ and its odd extension $f_o(x)$. Let w_0 be the solution to the wave equation over the entire real line with f_o as initial data:

$$\begin{aligned}
&\partial_{tt} w_o - 4\partial_{xx} w_o = 0, \quad x \in \mathbb{R},\ t > 0\\
&w_o(x,0) = f_o(x), \quad x > 0, \qquad \partial_t w_o(x,0) = 0, \quad x \in \mathbb{R}.
\end{aligned}$$

The solution to this problem is given by the D'Alembert formula

$$w_o(x,t) = \frac{1}{2}(f_o(x-2t) + f_o(x+2t)), \qquad \text{for all } x \in \mathbb{R} \text{ and } t \ge 0.$$

Let x be positive. Then $w(x,t) = w_o(x,t)$ (since by construction $w_o(0,t) = 0$ for all times).

Case 1: If $x - 2t > 0$, $f_o(x-2t) = f(x-2t)$; as a result

$$w(x,t) = \frac{1}{2}(f(x-2t) + f(x+2t).$$

Case 2: If $x - 2t < 0$, $f_o(x-2t) = -f(-x+2t)$; as a result

$$w(x,t) = \frac{1}{2}(-f(-x+2t) + f(x+2t)).$$

Note that actually $f_0(x) = x(1+x^2)^{-1}$; as a result, the solution can also be re-written as follows:

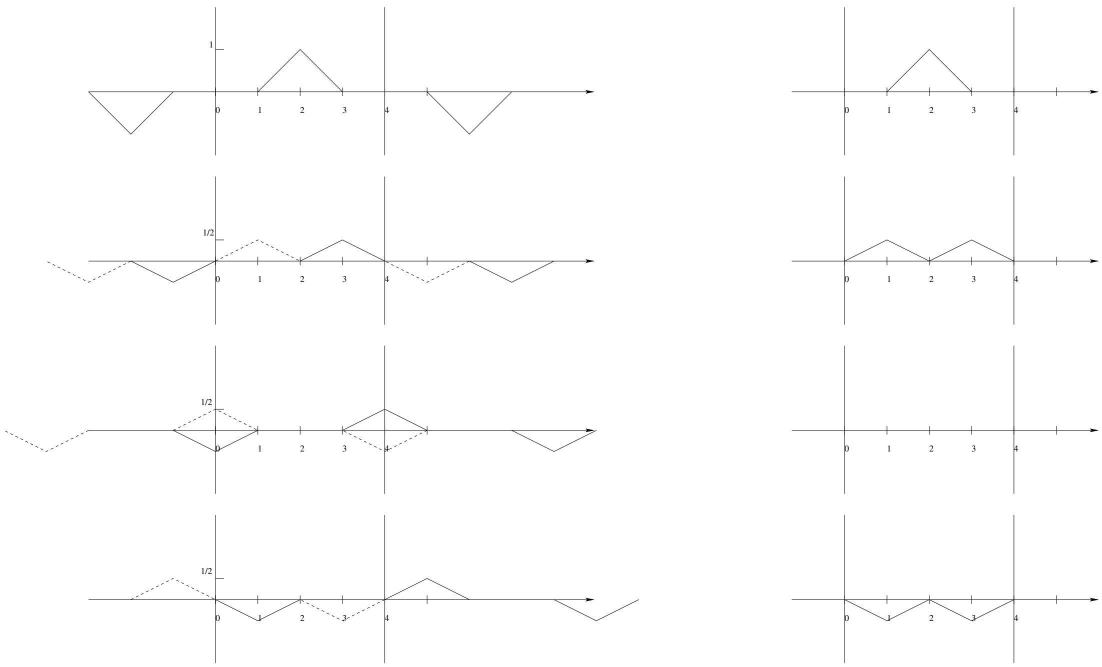
$$w(x,t) = \frac{1}{2}((x-2t)(1+(x-2t)^2)^{-1} + (x+2t)(1+(x+2t)^2)^{-1}).$$

Question 68: Consider the wave equation

$$\partial_{tt}w - \partial_{xx}w = 0, \quad x < 0,\ t > 0$$
$$w(x,0) = f(x), \quad x < 0, \qquad \partial_t w(x,0) = 0, \quad x < 0, \quad \text{and} \quad w(0,t) = 0, \quad t > 0.$$

where $f(x) = -x$, if $x \in [-1,0]$, $f(x) = 2 + x$, if $x \in [-2,-1]$, and $f(x) = 0$ otherwise. Give a graphical solution to the problem at $t = 0$, $t = 1$, and $t = 2$ (draw three different graphs and explain what you do)

Solution:



Initial data + odd extension Solution

Question 69: Solve the PDE

$$u_{tt} - a^2 u_{xx} = 0, \qquad -\infty < x < +\infty, \quad 0 \le t,$$
$$u(x,0) = \sin(x), \quad u_t(x,0) = a\cos(x), \qquad -\infty < x < +\infty.$$

Solution: Apply D'Alembert's Formula.

$$\begin{aligned} u(x,t) &= \frac{1}{2}(\sin(x+at) + \sin(x-at)) + \frac{1}{2a}\int_{x-at}^{x+at} a\cos(\xi)\xi \\ &= \frac{1}{2}(\sin(x+at) + \sin(x-at)) + \frac{1}{2}(\sin(x+at) - \sin(x-at)) \\ &= \sin(x+at). \end{aligned}$$

Question 70: Solve the PDE

$$u_{tt} - u_{xx} = 0, \qquad 0 \le x \le 1, \quad 0 < t,$$
$$u(0,t) = 0, \quad u(1,t) = 0 \qquad 0 < t,$$
$$u(x,0) = \sin(\pi x), \quad u_t(x,0) = 0, \qquad 0 < x < +\infty.$$

(Hint: Consider the periodic extension over $\mathbb{R}$ of the odd extension of u over $[-1+1]$).

Solution: The odd extension of u over $[-1+1]$, say u_o, satisfies the PDE and the initial conditions, and always satisfies $u_o(0,t) = 0$, $u_o(1,t) = 0$, $u_o(-1,t) = 0$. Since $u_o(1,t) = u_o(-1,t) = 0$, the periodic extension, says u_p, is smooth and also satisfies the PDE plus the initial conditions. As a result, we can obtain u by computing the solution of the wave equation on $\mathbb{R}$ using the periodic extension over $\mathbb{R}$ of the odd extension of the initial data over $[-1+1]$, i.e., $u = u_p|_{[0,1]}$

We have to define the odd extension of $\sin(\pi x)$ on $(-1,+1)$. Clearly $\sin(\pi x)$ is the odd extension. Now we define the periodic extension of $\sin(\pi x)$ over the entire real line. Clearly $\sin(\pi x)$ is the extension in question. The D'Alembert formula, which is valid on the entire real line, gives

$$\begin{aligned} u(x,t) &= \tfrac{1}{2}(\sin(\pi(x-t)) + \sin(\pi(x+t)) \\ &= \tfrac{1}{2}((\cos(\pi t)\sin(\pi x) - \sin(\pi t)\cos(\pi x)) + \tfrac{1}{2}((\cos(\pi t)\sin(\pi x) + \sin(\pi t)\cos(\pi x)) \\ &= \cos(\pi t)\sin(\pi x). \end{aligned}$$

Hence $u(x,t) = \cos(\pi t)\sin(\pi x)$ for all $x \in (0,1)$, $t > 0$.

Question 71: Solve $u_{tt} - 4u_{xx} = 0$, $x \in (-\infty,+\infty)$ and $t \geq 0$, with $u(x,0) = \sin^2(x)$, $\partial_t u(x,0) = -\frac{2x}{(1+x^2)^2}$.

Solution: The speed is 2. We apply the D'Alembert formula.

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left(\sin^2(x-2t) + \sin^2(x+2t)\right) + \frac{1}{4}\int_{x-2t}^{x+2t} -\frac{2\xi}{(1+\xi^2)^2}\mathrm{d}\xi, \\ &= \frac{1}{2}\left(\sin^2(x-2t) + \sin^2(x+2t)\right) + \frac{1}{4}\left(\frac{1}{1+\xi^2}\right|_{x-2t}^{x+2t} \\ &= \frac{1}{2}\left(\sin^2(x-2t) + \sin^2(x+2t)\right) + \frac{1}{4}\left(\frac{1}{1+(x+2t)^2} - \frac{1}{1+(x-2t)^2}\right). \end{aligned}$$

Question 72: Consider the wave equation $\partial_{tt}w - \partial_{xx}w = 0$, $x \in (0,4)$, $t > 0$, with

$$w(x,0) = f(x), \quad x \in (0,4), \quad \partial_t w(x,0) = 0, \quad x \in (0,4), \quad \text{and} \quad w(0,t) = 0, \quad w(4,t) = 0, \quad t > 0.$$

where $f(x) = x - 1$, if $x \in [1,2]$, $f(x) = 3 - x$, if $x \in [2,3]$, and $f(x) = 0$ otherwise. Give a simple expression of the solution in terms of an extension of f. Give a graphical solution to the problem at $t = 0$, $t = 1$, $t = 2$, and $t = 3$ (draw four different graphs and explain).

Solution: We know from class that with Dirichlet boundary conditions, the solution to this problem is given by the D'Alembert formula where f must be replaced by the periodic extension (of period 8) of its odd extension, say $f_{\mathsf{o,p}}$, where

$$f_{\mathsf{o,p}}(x+8) = f_{\mathsf{o,p}}(x), \qquad \forall x \in \mathbb{R}$$
$$f_{\mathsf{o,p}}(x) = \begin{cases} f(x) & \text{if } x \in [0,4] \\ -f(-x) & \text{if } x \in [-4,0) \end{cases}$$

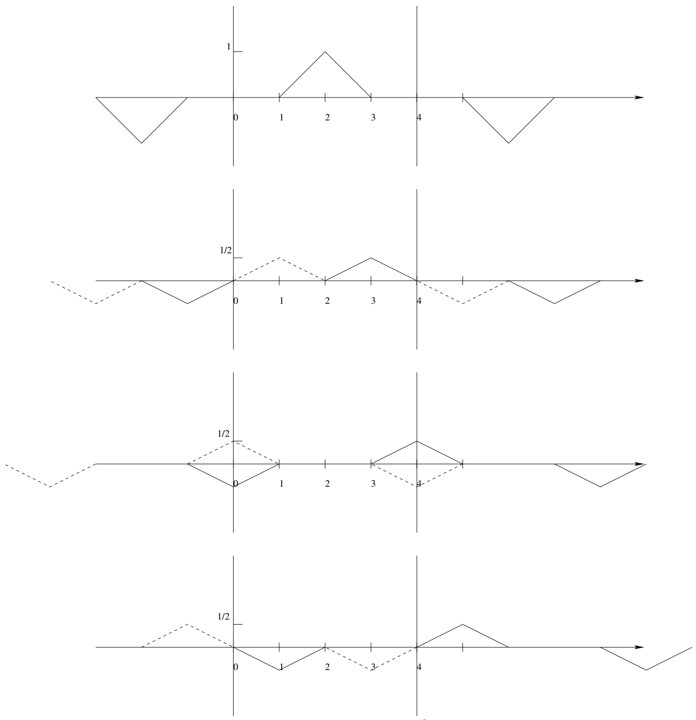
The solution is

$$u(x,t) = \frac{1}{2}(f_{\mathsf{o,p}}(x-t) + f_{\mathsf{o,p}}(x+t)).$$

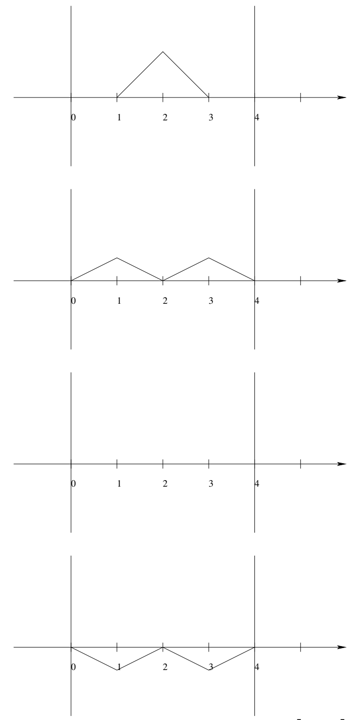
I draw on the left of the figure the graph of $f_{\mathsf{o,p}}$. Half the graph moves to the right with speed 1, the other half moves to the left with speed 1.

Question 73: Solve $u_{tt} - 4u_{xx} = 0$, $x \in (0,1)$ and $t \geq 0$, with $u(0,t) = u(1,t) = 0$, $u(x,0) = 0$, $\partial_t u(x,0) = g(x) := 2\pi\sin(\pi x)$. (Hint: use an extension technique).

Solution: We notice first that the wave speed is 2. We define g_o to be the odd extension of g over $(-1,+1)$. Clearly $g_o(x) = 2\pi\sin(\pi x)$ since $\sin(\pi x)$ is odd. We define g_{op} to be the periodic extension of g_o over $(\infty,+\infty)$ with period 2. Clearly, $g_{op}(x) = 2\pi\sin(\pi x)$ since 2 is a period for $\sin(\pi x)$. From class we know that the solution to the above problem is given by the restriction of



Initial data + periodic extension of the odd extension at $t = 0, 1, 2, 3$.



Solution in domain $[0, 4]$ at $t = 0, 1, 2, 3$

the D'Alembert formula to the interval $[0, 1]$:

$$
\begin{aligned}
u(x,t) &= \frac{1}{4}\int_{x-2t}^{x+2t} g_{op}(\xi)\mathsf{d}\xi = \frac{1}{4}\int_{x-2t}^{x+2t} 2\pi\sin(\pi\xi)\mathsf{d}\xi, \\
&= -\frac{1}{2}\left(\cos(\pi(x+2t)) - \cos(\pi(x-2t))\right) \\
&= \sin(\frac{\pi}{2}(2x))\sin(\frac{\pi}{2}(4t)) \\
&= \sin(\pi x)\sin(2\pi t), \qquad \forall x \in [0,1], \forall t \geq 0.
\end{aligned}
$$

Question 74: Solve $u_{tt} - 4u_{xx} = 0$, $x \in (0,1)$ and $t \geq 0$, with $u(0,t) = 0$, $\partial_x u(1,t) = 0$, $u(x,0) = 0$, $\partial_t u(x,0) = g(x) := 2\pi\sin(\frac{\pi}{2}x)$. (Hint: Pay attention to the boundary conditions. Use <u>three</u> extensions.)

Solution: To be able to apply the d'Alembert formula, we need to extend the above problem to the $(-\infty, +\infty)$. the Dirichlet condition a $x = 0$ requires an odd extension and the Neumann condition requires an even extension.

We define g_o to be the odd extension of g over $(-1, +1)$ to account for the Dirichlet boundary condition at $x = 0$.

$$
g_o(x) = \begin{cases} g(x) & \text{for all } x \in (0,1), \\ -g(-x) & \text{for all } x \in (-1,0) \end{cases}
$$

Clearly $g_o(x) = 2\pi\sin(\frac{\pi}{2}x)$ since $\sin(\frac{\pi}{2}x)$ is odd. More precisely,

$$
g_o(x) = \begin{cases} g(x) = 2\pi\sin(\frac{\pi}{2}x) & \text{for all } x \in (0,1), \\ -g(-x) = -2\pi\sin(\frac{\pi}{2}(-x)) = 2\pi\sin(\frac{\pi}{2}x) & \text{for all } x \in (-1,0). \end{cases}
$$

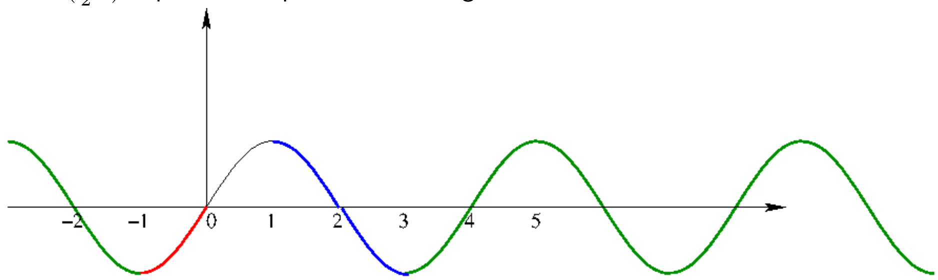
Now we need to consider the even extension of g_o about the point $x = 1$ to account for the Neumann boundary condition at $x = 1$. Let us denote $g_{oe}(x)$ this extension. The function $g_{oe}(x)$ is such that

$$
g_{oe}(x) = \begin{cases} g_{oe}(x) = g_o(x) & \text{for all } x \in (-1,1), \\ g_{oe}(x) = g_o(2-x) & \text{for all } x \in (1,3) \end{cases}
$$

Now we observe that $\sin(\frac{\pi}{2}(2-x)) = \sin(\pi - \frac{\pi}{2}x) = \sin(\frac{\pi}{2}x)$, which means that $g_{oe}(x) = 2\pi\sin(\frac{\pi}{2}x)$. More precisely,

$$g_{oe}(x) = \begin{cases} g_{oe}(x) = g_o(x) = 2\pi\sin(\frac{\pi}{2}x) & \text{for all } x \in (-1,1), \\ g_{oe}(x) = g_o(2-x) = 2\pi\sin(\frac{\pi}{2}(2-x)) = 2\pi\sin(\frac{\pi}{2}x) & \text{for all } x \in (1,3). \end{cases}$$

Now we consider the periodic extension of g_{oe} of period 4, say g_{oep}. Clearly $g_{oep} = 2\pi\sin(\frac{\pi}{2}x)$, since $42\pi\sin(\frac{\pi}{2}x)$ is periodic of period 4. See Figure



We notice finally that the wave speed is 2. From class we know that the solution to the above problem is given by the restriction of the D'Alembert formula to the interval $[0,1]$:

$$\begin{aligned} u(x,t) &= \frac{1}{4}\int_{x-2t}^{x+2t} g_{oep}(\xi)\mathrm{d}\xi = \frac{1}{4}\int_{x-2t}^{x+2t} 2\pi\sin(\frac{\pi}{2}\xi)\mathrm{d}\xi, \\ &= -\Big(\cos(\frac{\pi}{2}(x+2t)) - \cos(\frac{\pi}{2}(x-2t))\Big) \\ &= 2\sin(\frac{\pi}{2}(x))\sin(\frac{\pi}{2}(2t)) \\ &= 2\sin(\frac{\pi}{2}x)\sin(\pi t), \qquad \forall x \in [0,1], \forall t \geq 0. \end{aligned}$$

Question 75: Consider the wave equation $\partial_{tt}w - \partial_{xx}w = 0$, $x \in (-\infty,+\infty)$, $t > 0$, with initial data $u(x,0) = \frac{1}{1+x^2}$, $\partial_t u(x,0) = \frac{2x}{(1+x^2)^2}$. Compute the solution $w(x,t)$.

Solution: The wave speed is 2. The solution is given by the D'Alembert formula,

$$w(x,t) = \frac{1}{2}\left(\frac{1}{1+(x-t)^2} + \frac{1}{1+(x+t)^2}\right) + \frac{1}{2}\int_{x-t}^{x+t} \frac{2\tau}{(1+\tau^2)^2}\mathrm{d}\tau$$

After integration, we obtain

$$= \frac{1}{2}\left(\frac{1}{1+(x-t)^2} + \frac{1}{1+(x+t)^2}\right) - \frac{1}{2}\left[\frac{1}{(1+\tau^2)}\right]_{x-t}^{x+t},$$

which finally gives

$$w(x,t) = \frac{1}{1+(x-t)^2}.$$

Question 76: Consider the wave equation $\partial_{tt}w - \partial_{xx}w = 0$, $x \in (0,4)$, $t > 0$, with

$$w(x,0) = f(x), \quad x \in (0,4), \quad \partial_t w(x,0) = 0, \quad x \in (0,4), \quad \text{and} \quad \partial_x w(0,t) = 0, \quad \partial_x w(4,t) = 0, \quad t > 0.$$

where $f(x) = x - 1$, if $x \in [1,2]$, $f(x) = 3 - x$, if $x \in [2,3]$, and $f(x) = 0$ otherwise. Give a simple expression of the solution in terms of an extension of f. Give a graphical solution to the problem at $t = 0$, $t = 1$, $t = 2$, and $t = 3$ (draw four different graphs and explain).

Solution: We know from class that with homogeneous Neumann boundary conditions, the solution to this problem is given by the D'Alembert formula where f must be replaced by the periodic extension (of period 8) of its even extension, say $f_{\mathsf{e,p}}$, where

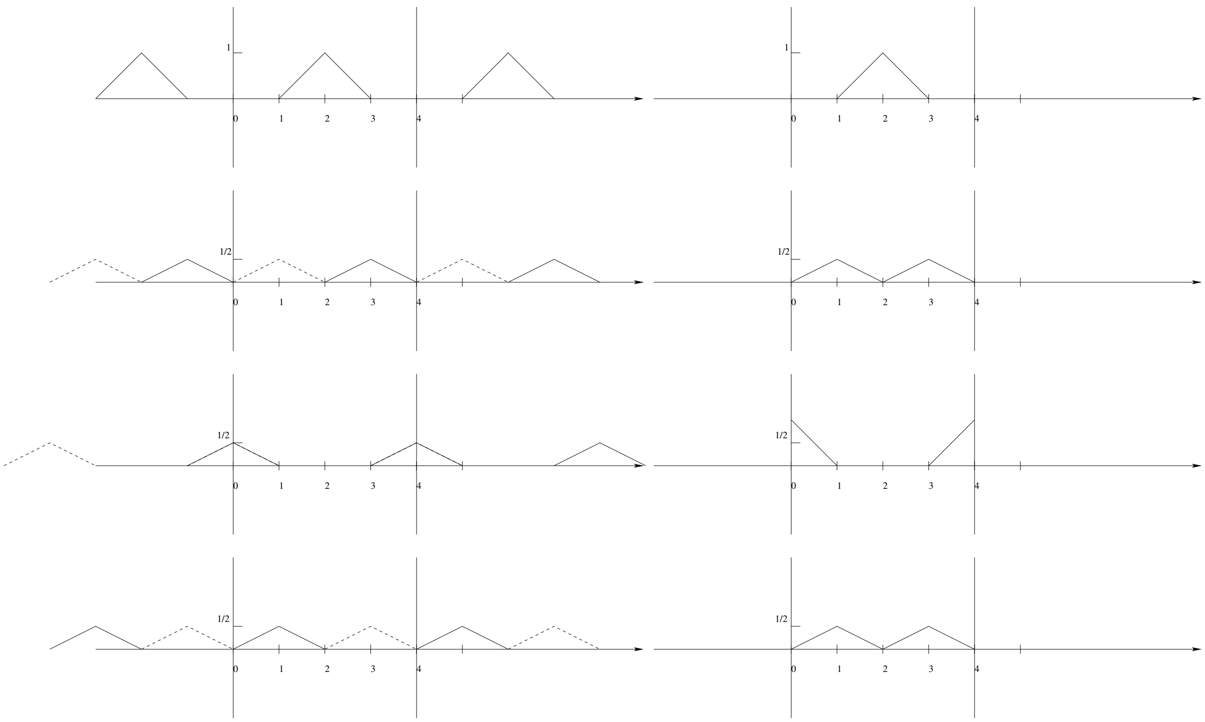
$$f_{\mathsf{e,p}}(x+8) = f_{\mathsf{e,p}}(x), \qquad \forall x \in \mathbb{R}$$

$$f_{\mathsf{e,p}}(x) = \begin{cases} f(x) & \text{if } x \in [0,4] \\ f(-x) & \text{if } x \in [-4,0) \end{cases}$$

The solution is

$$u(x,t) = \frac{1}{2}(f_{\mathsf{e,p}}(x-t) + f_{\mathsf{e,p}}(x+t)).$$

I draw on the left of the figure the graph of $f_{\mathsf{o,p}}$. Half the graph moves to the right with speed 1, the other half moves to the left with speed 1.



(a) Initial data + periodic extension of the even extension at $t = 0, 1, 2, 3$. Solid line waves move to the right, dotted line waves move to the left

(b) Solution in domain $(0, 4)$ at $t = 0, 1, 2, 3$

Question 77: Solve $u_{tt} - 4u_{xx} = 0$, $x \in (0,1)$ and $t \geq 0$, with $u(0,t) = 0$, $\partial_x u(1,t) = 0$, $u(x,0) = 0$, $\partial_t u(x,0) = g(x) := 2\pi \sin(\frac{\pi}{2}x)$. (Hint: Pay attention to the boundary conditions. Use <u>three</u> extensions.)

Solution: To be able to apply the d'Alembert formula, we need to extend the above problem to the $(-\infty, +\infty)$. the Dirichlet condition a $x = 0$ requires an odd extension and the Neumann condition requires an even extension.

We define g_o to be the odd extension of g over $(-1, +1)$ to account for the Dirichlet boundary condition at $x = 0$.

$$g_o(x) = \begin{cases} g(x) & \text{for all } x \in (0,1), \\ -g(-x) & \text{for all } x \in (-1,0) \end{cases}$$

Clearly $g_o(x) = 2\pi \sin(\frac{\pi}{2}x)$ since $\sin(\frac{\pi}{2}x)$ is odd. More precisely,

$$g_o(x) = \begin{cases} g(x) = 2\pi \sin(\frac{\pi}{2}x) & \text{for all } x \in (0,1), \\ -g(-x) = -2\pi \sin(\frac{\pi}{2}(-x)) = 2\pi \sin(\frac{\pi}{2}x) & \text{for all } x \in (-1,0). \end{cases}$$

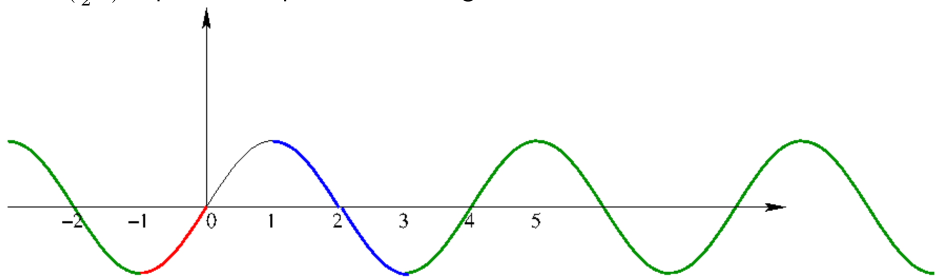
Now we need to consider the even extension of g_o about the point $x = 1$ to account for the Neumann boundary condition at $x = 1$. Let us denote $g_{oe}(x)$ this extension. The function $g_{oe}(x)$ is such that

$$g_{oe}(x) = \begin{cases} g_{oe}(x) = g_o(x) & \text{for all } x \in (-1,1), \\ g_{oe}(x) = g_o(2-x) & \text{for all } x \in (1,3) \end{cases}$$

Now we observe that $\sin(\frac{\pi}{2}(2-x)) = \sin(\pi - \frac{\pi}{2}x) = \sin(\frac{\pi}{2}x)$, which means that $g_{oe}(x) = 2\pi\sin(\frac{\pi}{2}x)$. More precisely,

$$g_{oe}(x) = \begin{cases} g_{oe}(x) = g_o(x) = 2\pi\sin(\frac{\pi}{2}x) & \text{for all } x \in (-1,1), \\ g_{oe}(x) = g_o(2-x) = 2\pi\sin(\frac{\pi}{2}(2-x)) = 2\pi\sin(\frac{\pi}{2}x) & \text{for all } x \in (1,3). \end{cases}$$

Now we consider the periodic extension of g_{oe} of period 4, say g_{oep}. Clearly $g_{oep} = 2\pi\sin(\frac{\pi}{2}x)$, since $42\pi\sin(\frac{\pi}{2}x)$ is periodic of period 4. See Figure



We notice finally that the wave speed is 2. From class we know that the solution to the above problem is given by the restriction of the D'Alembert formula to the interval $[0,1]$:

$$\begin{aligned} u(x,t) &= \frac{1}{4}\int_{x-2t}^{x+2t} g_{oep}(\xi)\mathrm{d}\xi = \frac{1}{4}\int_{x-2t}^{x+2t} 2\pi\sin(\frac{\pi}{2}\xi)\mathrm{d}\xi, \\ &= -\left(\cos(\frac{\pi}{2}(x+2t)) - \cos(\frac{\pi}{2}(x-2t))\right) \\ &= 2\sin(\frac{\pi}{2}(x))\sin(\frac{\pi}{2}(2t)) \\ &= 2\sin(\frac{\pi}{2}x)\sin(\pi t), \qquad \forall x \in [0,1], \forall t \geq 0. \end{aligned}$$

Question 78: Consider the wave equation $\partial_{tt}w - 4\partial_{xx}w = 0$, $x \in (-\infty,+\infty)$, $t > 0$, with initial data $u(x,0) = \frac{1}{1+x^2}$, $\partial_t u(x,0) = -\frac{4x}{(1+x^2)^2}$. Compute the solution $w(x,t)$.

Solution: The wave speed is 2. The solution is given by the D'Alembert formula,

$$w(x,t) = \frac{1}{2}\left(\frac{1}{1+(x-2t)^2} + \frac{1}{1+(x+2t)^2}\right) + \frac{1}{4}\int_{x-2t}^{x+2t} -\frac{4\tau}{(1+\tau^2)^2}\mathrm{d}\tau$$

After integration, we obtain

$$= \frac{1}{2}\left(\frac{1}{1+(x-2t)^2} + \frac{1}{1+(x+2t)^2}\right) + \frac{1}{4}\left[\frac{2}{(1+\tau^2)}\right]_{x-2t}^{x+2t},$$

which finally gives

$$w(x,t) = \frac{1}{1+(x+2t)^2}.$$

Question 79: Consider the wave equation $\partial_{tt}w - \partial_{xx}w = 0$, $x \in (0,4)$, $t > 0$, with

$$w(x,0) = f(x), \quad x \in (0,4), \quad \partial_t w(x,0) = 0, \quad x \in (0,4), \quad \text{and} \quad \partial_x w(0,t) = 0, \quad \partial_x w(4,t) = 0, \quad t > 0.$$

where $f(x) = x - 1$, if $x \in [1,2]$, $f(x) = 3 - x$, if $x \in [2,3]$, and $f(x) = 0$ otherwise. Give a simple expression of the solution in terms of an extension of f. Give a graphical solution to the problem at $t = 0$, $t = 1$, $t = 2$, and $t = 3$ (draw four different graphs and explain).

Solution: We know from class that with homogeneous Neumann boundary conditions, the solution to this problem is given by the D'Alembert formula where f must be replaced by the periodic extension (of period 8) of its even extension, say $f_{\mathsf{e,p}}$, where

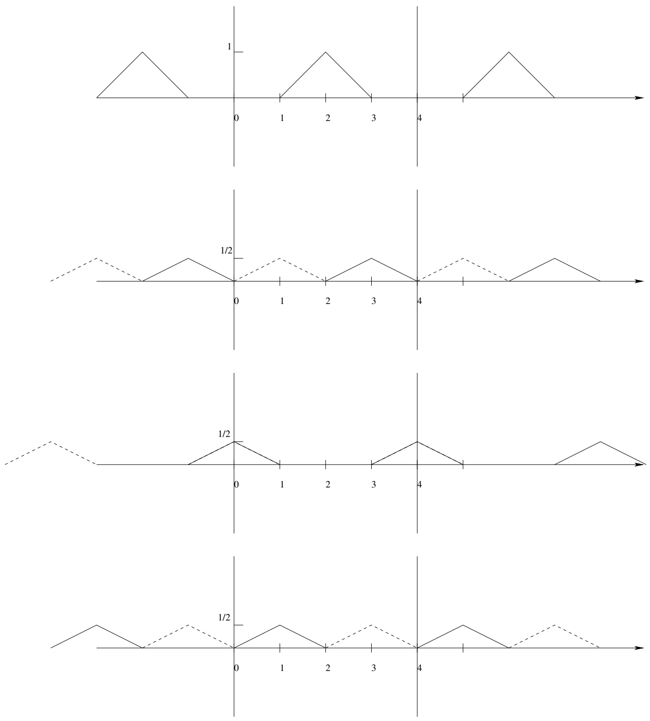
$$f_{\mathsf{e,p}}(x+8) = f_{\mathsf{e,p}}(x), \qquad \forall x \in \mathbb{R}$$

$$f_{\mathsf{e,p}}(x) = \begin{cases} f(x) & \text{if } x \in [0,4] \\ f(-x) & \text{if } x \in [-4,0) \end{cases}$$

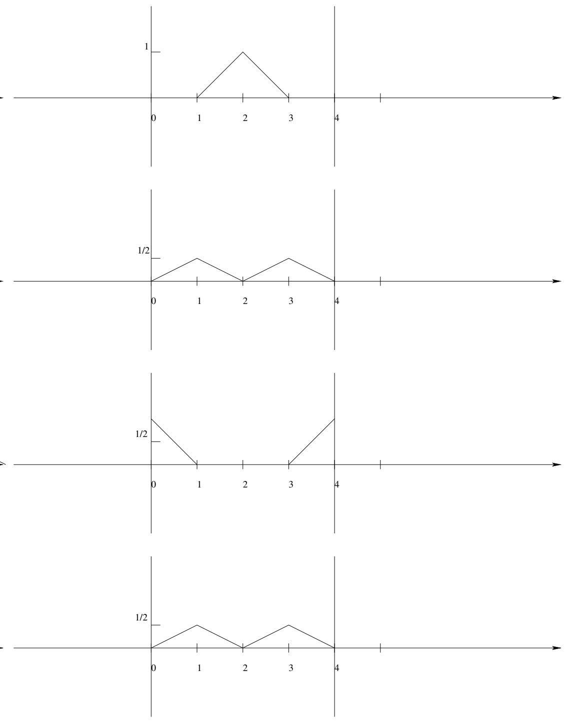
The solution is

$$u(x,t) = \frac{1}{2}(f_{\mathsf{e,p}}(x-t) + f_{\mathsf{e,p}}(x+t)).$$

I draw on the left of the figure the graph of $f_{\mathsf{o,p}}$. Half the graph moves to the right with speed 1, the other half moves to the left with speed 1.



(c) Initial data + periodic extension of the even extension at $t = 0, 1, 2, 3$. Solid line waves move to the right, dotted line waves move to the left



(d) Solution in domain $(0, 4)$ at $t = 0, 1, 2, 3$

6 Method of characteristics

Question 80: (a) Show that the PDE $u_y = 0$ in the half plane $\{x > 0\}$ has no solution which is $\mathcal{C}^1$ and satisfies the boundary condition $u(y^2, y) = y$.

Solution: The PDE implies that $u(x, y) = \phi(x)$ where ϕ is any $\mathcal{C}^1$ function. The boundary condition implies $\phi(1) = u(1, -1) = -1$ and $\phi(1) = u(1, 1) = 1$, which is impossible. The reason for this happening is that the characteristics lines ($x = c$) cross the boundary curve (the parabola of equation $x = y^2$) twice.

(b) Find the $\mathcal{C}^1$ function that solves the above PDE in the quadrant $\{x > 0, 0 > y\}$ (beware the sign of y).

Solution: The PDE implies $u(x, y) = \phi(x)$ and the boundary condition implies $\phi(y^2) = u(y^2, y) = y = -|y|$ since y is negative. Then $u(x, y) = \phi(x) = -\sqrt{x}$.

Question 81: Let $\Omega = \{x > 0, y > 0\}$ be the first quadrant of the plane. Let Γ be the line defined by the following parameterization $\Gamma = \{x = s, y = 1/s, s > 0\}$. Solve the following PDE:

$$\begin{aligned} &xu_x + 2yu_y = 0, \quad \text{in } \Omega, \\ &u(x, y) = x \quad \text{on } \Gamma. \end{aligned}$$

Solution: The characteristics are $X(\tau, s) = se^{\tau}$, $Y(\tau, s) = s^{-1}e^{2\tau}$. Upon setting $u(X(\tau, s), Y(\tau, s)) = w(\tau, s)$, we obtain $w(\tau, s) = w(0, s)$. Then the boundary condition implies $w(0, s) = u(s, \frac{1}{s}) = s$. In other words $u(x, y) = (x^2y^{-1})^{1/3}$.

Question 82: (a) Solve the quasi-linear PDE $3u^2u_x + 3u^2u_y = 1$ in the plane by using the method of Lagrange (that is, show that u solves the nonlinear equation $c(a(x, y, u), b(x, y, u)) = 0$ where c is an arbitrary function and a, b are polynomials of degree 3 that you must find.)

Solution: The auxiliary equation is $3z^2\phi_x + 3z^2\phi_y + \phi_z = 0$. Define the plane $\Gamma = \{x = s, y = s', z = 0\}$ and enforce $\phi(x, y, z) = \phi_0(s, s')$ on Γ, where ϕ_0 is an arbitrary $\mathcal{C}^1$ function. The characteristics are $X(\tau, s, s') = \tau^3 + s$ $Y(\tau, s, s') = \tau^3 + s'$, $Z(\tau, s, s') = \tau$. Then $\phi(x, y, z) = \phi_0(s, s')$ where $s = x - z^3$ and $s' = y - z^3$. Then $\phi(x, y, z) = \phi_0(x - z^3, y - z^3)$. Hence, u solves $\phi_0(x - u^3, y - u^3) = 0$.

(b) Find a solution to the above PDE that satisfies the boundary condition $u(x, 2x) = 1$.

Solution: We want $\phi_0(x-1, 2x-1) = 0$. Take $\phi_0(\alpha, \beta) = 2\alpha - \beta + 1$. Then $2(x-u^3) - (y-u^3) + 1 = 0$, that is $u(x, y) = (1 + 2x - y)^{1/3}$.

Question 83: We want to solve the following PDE:

$$\begin{aligned}
&\partial_t w + 3\partial_x w = 0, \quad x > -t,\ t > 0\\
&w(x,t) = w_\Gamma(x,t), \text{ for all } (x,t) \in \Gamma \text{ where}\\
&\Gamma = \{(x,t) \in \mathbb{R}^2 \text{ s.t. } x = -t,\ x < 0\} \cup \{(x,t) \in \mathbb{R}^2 \text{ s.t. } t = 0,\ x \geq 0\}\\
&\text{and } w_\Gamma \text{ is a given function.}
\end{aligned}$$

(a) Draw a picture of the domain Ω where the PDE must be solved, of the boundary Γ, and of the characteristics.

Solution:

(b) Define a one-to-one parametric representation of the boundary Γ.

Solution: For negative s we set $x_\Gamma(s) = s$ and $t_\Gamma(s) = -s$; clearly we have $x_\Gamma(s) = -t_\Gamma(s)$ for all $s < 0$. For positive s we set $x_\Gamma(s) = s$ and $t_\Gamma(s) = 0$. The map $\mathbb{R} \in s \mapsto (x_\Gamma(s), t_\Gamma(s)) \in \Gamma$ is one-t-one.

(c) Give a parametric representation of the characteristics associated with the PDE.

Solution: (i) We use t and s to parameterize the characteristics. The characteristics are defined by

$$\partial_t X(t,s) = 3, \quad \text{with} \quad x(t_)\Gamma(s), s) = x_\Gamma(s).$$

This yields the following parametric representation of the characteristics

$$X(t,s) = 3(t - t_\Gamma(s)) + x_\Gamma(s),$$

where $t \geq 0$ and $s \in (-\infty, +\infty)$.

(d) Give an implicit parametric representation of the solution to the PDE.

Solution: (i) Now we set $\phi(t,s) = w(X(t,s), t(t,s))$ and we insert this ansatz in the equation. This gives $\frac{d\phi}{dt}(t,s) = 0$, i.e., $\phi(t,s)$ does not depend on t. In other words

$$w(X(t,s), t(t,s)) = \phi(t,s) = \phi(0,s) = w(x(0,s), t(0,s)) = w_\Gamma(x_\Gamma(s), t_\Gamma(s))$$

A parametric representation of the solution is given by

$$\begin{aligned}
&X(t,s) = 3(t - t_\Gamma(s)) + x_\Gamma(s),\\
&w(X(t,s), t(t,s)) = w_\Gamma(x_\Gamma(s), t_\Gamma(s)).
\end{aligned}$$

(e) Give an explicit representation of the solution.

Solution: (i) We have to find the inverse map $(x,t) \mapsto (t,s)$. Clearly $x - 3t = x_\Gamma(s) - 3t_\Gamma(s)$. Then, there are two cases depending on the sign of s.
case 1: If $s < 0$, then $x_\Gamma(s) = s$ and $t_\Gamma(s) = -s$. That means $x - 3t = 4s$, which in turns implies $s = \frac{1}{4}(x - 3t)$. Then

$$w(x,t) = w_\Gamma(\tfrac{1}{4}(x - 3t), -\tfrac{1}{4}(x - 3t)), \quad \text{if } x - 3t < 0.$$

case 2: If $s \geq 0$, then $x_\Gamma(s) = s$ and $t_\Gamma(s) = 0$. That means $x - 3t = s$. Then

$$w(x,t) = w_\Gamma(x - 3t, 0), \quad \text{if } x - 3t \geq 0.$$

Note that the explicit representation of the solution does not depend on the choice of the parameterization.

Question 84: Solve the following PDE by the method of characteristics:

$$\begin{aligned}
&\partial_t w + 3\partial_x w = 0, \quad x > 0,\ t > 0\\
&w(x,0) = f(x), \quad x > 0, \quad \text{and} \quad w(0,t) = h(t), \quad t > 0.
\end{aligned}$$

Solution: First we parameterize the boundary of Ω by setting $\Gamma = \{x = x_\Gamma(s), t = t_\Gamma(s);\ s \in \mathbb{R}\}$ with

$$x_\Gamma(s) = \begin{cases} 0 & \text{if } s < 0, \\ s, & \text{if } s \geq 0. \end{cases} \quad \text{and} \quad t_\Gamma(s) = \begin{cases} -s & \text{if } s < 0, \\ 0, & \text{if } s \geq 0. \end{cases}$$

The we define the characteristics by

$$\partial_t X(s,t) = 3, \quad \text{with} \quad X(s, t_\Gamma(s)) = x_\Gamma(s).$$

The general solution is $X(s,t) = 3(t - t_\Gamma(s)) + x_\Gamma(s)$. Now we make the change of variable $\phi(s,t) = w(X(s,t),t)$ and we compute $\partial_t\phi(s,t)$,

$$\partial_t\phi(s,t) = \partial_t w(X(s,t),t) + \partial_x w(X(s,t),t)\partial_t X(s,t) = \partial_t w(X(s,t),t) + 3\partial_x w(X(s,t),t) = 0.$$

This means that $\phi(s,t) = \phi(s, t_\Gamma(s))$. In other words

$$w(X(s,t),t) = w(X(s,t_\Gamma(s)), t_\Gamma(s)) = w(x_\Gamma(s), t_\Gamma(s)).$$

<u>Case 1:</u> If $s < 0$, then $X(s,t) = 3(t - t_\Gamma(s))$. This implies $t_\Gamma(s) = t - X/3$. The condition $s < 0$ and the definition $t_\Gamma(s) = -s$ imply $t - X/3 \geq 0$. Moreover we have

$$w(X,t) = w(0, t_\Gamma(s)) = h(t_\Gamma(s)).$$

In conlusion

$$w(X,t) = h(t - X/3), \quad \text{if} \quad 3t > X.$$

<u>Case 2:</u> If $s \geq 0$, then $X(s,t) = 3t + x_\Gamma(s)$. This implies $x_\Gamma(s) = X - 3t$. The condition $s \geq 0$ and the definition $x_\Gamma(s) = s$ imply $X - 3t \geq 0$. Moreover we have

$$w(X,t) = w(x_\Gamma(s), 0) = f(x_\Gamma(s)).$$

In conlusion

$$w(X,t) = f(X - 3t), \quad \text{if} \quad X \geq 3t.$$

Question 85: Let $\Omega = \{(x,t) \in \mathbb{R}^2;\ x + 2t \geq 0\}$. Solve the following PDE in explicit form with the method of characteristics:

$$\partial_t u(x,t) + 3\partial_x u(x,t) = u(x,t), \quad \text{in } \Omega, \quad \text{and} \quad u(x,t) = 1 + \sin(x), \text{ if } x + 2t = 0.$$

Solution: (i) First we parameterize the boundary of Ω by setting $\Gamma = \{x = x_\Gamma(s), t = t_\Gamma(s);\ s \in \mathbb{R}\}$ with $x_\Gamma(s) = -2s$ and $t_\Gamma(s) = s$. This choice implies

$$u(x_\Gamma(s), t_\Gamma(s)) := u_\Gamma(s) := 1 + \sin(-2s).$$

(ii) We compute the characteristics

$$\partial_t X(t,s) = 3, \quad X(t_\Gamma(s), s) = x_\Gamma(s).$$

The solution is $X(t,s) = 3(t - t_\Gamma(s)) + x_\Gamma(s)$.

(iii) Set $\Phi(t,s) := u(X(t,s),t)$ and compute $\partial_t\Phi(t,s)$. This gives

$$\begin{aligned}\partial_t\Phi(t,s) &= \partial_t u(X(t,s),t) + \partial_x u(X(t,s),t)\partial_t X(t,s) \\ &= \partial_t u(X(t,s),t) + 3\partial_x u(X(t,s),t) = u(X(t,s),t) = \Phi(t,s).\end{aligned}$$

The solution is $\Phi(t,s) = \Phi(t_\Gamma(s), s)e^{t - t_\Gamma(s)}$.

(iv) The implicit representation of the solution is

$$X(t,s) = 3(t - t_\Gamma(s)) + x_\Gamma(s) \quad u(X(t,s)) = u_\Gamma(s)e^{t - t_\Gamma(s)}.$$

(v) The explicit representation is obtained by using the definitions of $-t_\Gamma(s)$, $x_\Gamma(s)$ and $u_\Gamma(s)$.

$$X(s,t) = 3(t-s) - 2s = 3t - 5s,$$

which gives

$$s = \frac{1}{5}(3t - X).$$

The solution is

$$\begin{aligned} u(x,t) &= (1+\sin(\tfrac{2}{5}(x-3t)))e^{t-\frac{1}{5}(3t-x)} \\ &= (1+\sin(\tfrac{2(x-3t)}{5}))e^{\frac{x+2t}{5}}. \end{aligned}$$

Question 86: Let $\Omega = \{(x,t) \in \mathbb{R}^2;\ x \geq 0,\ t \geq 0\}$. Solve the following PDE in explicit form

$$\partial_t u(x,t) + t\partial_x u(x,t) = 2u(x,t), \quad \text{in } \Omega, \quad \text{and} \quad u(0,t) = t,\ u(x,0) = x.$$

Solution: (i) First we parameterize the boundary of Ω by setting $\Gamma = \{x = x_\Gamma(s), t = t_\Gamma(s);\ s \in \mathbb{R}\}$ with $x_\Gamma(s) = s$ and $t_\Gamma(s) = 0$ if $s > 0$ and $x_\Gamma(s) = 0$ and $t_\Gamma(s) = -s$ if $s \leq 0$. This choice implies

$$u(x_\Gamma(s), t_\Gamma(s)) := u_\Gamma(s) := \begin{cases} s & \text{if } s > 0 \\ -s & \text{if } s \leq 0 \end{cases}.$$

(ii) We compute the characteristics

$$\partial_t X(t,s) = t, \quad X(t_\Gamma(s), s) = x_\Gamma(s).$$

The solution is $X(t,s) = \frac{1}{2}t^2 - \frac{1}{2}t_\Gamma^2(s) + x_\Gamma(s)$.

(iii) Set $\Phi(t,s) := u(X(t,s),t)$ and compute $\partial_t\Phi(t,s)$. This gives

$$\begin{aligned} \partial_t\Phi(t,s) &= \partial_t u(X(t,s),t) + \partial_x u(X(t,s),t)\partial_t X(t,s) \\ &= \partial_t u(X(t,s),t) + t\partial_x u(X(t,s),t) = 2u(X(t,s),t) = 2\Phi(t,s). \end{aligned}$$

The solution is $\Phi(t,s) = \Phi(t_\Gamma(s), s)e^{2(t-t_\Gamma(s))}$.

(iv) The implicit representation of the solution is

$$X(t,s) = \frac{1}{2}t^2 - \frac{1}{2}t_\Gamma^2(s) + x_\Gamma(s), \quad u(X(t,s)) = u_\Gamma(s)e^{2(t-t_\Gamma(s))}, \quad u_\Gamma(s) = \begin{cases} s & \text{if } s > 0 \\ -s & \text{if } s \leq 0 \end{cases}.$$

(v) We distinguish two cases to get the explicit form of the solution:
Case 1: Assume $s > 0$, then $t_\Gamma(s) = 0$ and $x_\Gamma(s) = s$. This implies $X(t,s) = \frac{1}{2}t^2 + s$, meaning $s = X - \frac{1}{2}t^2$. The solution is

$$u(x,t) = (x - \frac{1}{2}t^2)e^{2t}, \quad \text{if} \quad x > \frac{1}{2}t^2.$$

Case 2: Assume $s \leq 0$, then $t_\Gamma(s) = -s$ and $x_\Gamma(s) = 0$. This implies $X(t,s) = \frac{1}{2}t^2 - \frac{1}{2}s^2$, meaning $s = -\sqrt{t^2 - 2X}$. The solution is

$$u(x,t) = \sqrt{t^2 - 2x}\, e^{2(t-\sqrt{t^2-2x})}, \quad \text{if} \quad x \leq \frac{1}{2}t^2.$$

Question 87: Let $\Omega = \{(t,x) \in \mathbb{R}^2 :\ t > 0,\ x \geq t\}$. Let Γ be defined by the following parameterization $\Gamma = \{x = x_\Gamma(s), t = t_\Gamma(s), s \in \mathbb{R}\}$, with $x_\Gamma(s) = -s$ and $t_\Gamma(s) = -s$ if $s \leq 0$,

$x_\Gamma(s) = s$ and $t_\Gamma(s) = 0$ if $s \geq 0$. Solve the following PDE (give the implicit and explicit representations):

$$u_t + 3u_x + 2u = 0, \quad \text{in } \Omega, \qquad u(x,t) = u_\Gamma(x,t) := \begin{cases} 1 & \text{if } t = 0 \\ 2 & \text{if } x = t \end{cases} \qquad \text{for all } (x,t) \text{ in } \Gamma.$$

Solution: We define the characteristics by

$$\frac{dx(t,s)}{dt} = 3, \quad x(t_\Gamma(s), s) = x_\Gamma(s).$$

This gives $x(t,s) = x_\Gamma(s) + 3(t - t_\Gamma(s))$. Upon setting $\phi(t,s) = u(x(t,s),t)$, we observe that $\partial_t \phi(t,s) + 2\phi(t,s) = 0$, which means

$$\phi(t,s) = ce^{-2t}.$$

The initial condition implies $\phi(t_\Gamma(s), s) = u_\Gamma(x_\Gamma(s), t_\Gamma(s))$; as a result $c = u_\Gamma(x_\Gamma(s), t_\Gamma(s))e^{2t_\Gamma(s)}$.

$$\phi(t,s) = u_\Gamma(x_\Gamma(s), t_\Gamma(s))e^{2(t_\Gamma(s)-t)}.$$

The implicit representation of the solution is

$$u(x(t,s),t) = u_\Gamma(x_\Gamma(s), t_\Gamma(s))e^{2(t_\Gamma(s)-t)}, \qquad x(t,s) = x_\Gamma(s) + 3(t - t_\Gamma(s)).$$

Now we give the explicit representation.
Case 1: If $s \leq 0$, $x_\Gamma(s) = -s$, $t_\Gamma(s) = -s$, and $u_\Gamma(x_\Gamma(s), t_\Gamma(s)) = 2$. This means $x(t,s) = -s + 3(t+s)$ and we obtain $s = \frac{1}{2}(x - 3t)$, which means

$$u(x,t) = 2e^{-2(\frac{1}{2}(x-3t)-t)} = 2e^{t-x}, \qquad \text{if } x - 3t < 0.$$

Case 2: If $s \geq 0$, $x_\Gamma(s) = s$, $t_\Gamma(s) = 0$, and $u_\Gamma(x_\Gamma(s), t_\Gamma(s)) = 1$. This means $x(t,s) = s + 3t$ and we obtain $s = x - 3t$, which means

$$u(x,t) = e^{-2t}, \qquad \text{if } x - 3t > 0.$$

Question 88: Let $\Omega = \{(t,x) \in \mathbb{R}^2 : \ t > 0, \ x \geq -\sqrt{t}\}$. Let Γ be defined by the following parameterization $\Gamma = \{x = x_\Gamma(s), t = t_\Gamma(s), s \in \mathbb{R}\}$, with $x_\Gamma(s) = s$ and $t_\Gamma(s) = s^2$ if $s \leq 0$, $x_\Gamma(s) = s$ and $t_\Gamma(s) = 0$ if $s \geq 0$. Solve the following PDE (give the implicit and explicit representations):

$$u_t + 2u_x + 3u = 0, \quad \text{in } \Omega, \qquad \text{and} \quad u(x_\Gamma(s), t_\Gamma(s)) := e^{-t_\Gamma(s) - x_\Gamma(s)}, \quad \forall s \in (-\infty, +\infty).$$

Solution: We define the characteristics by

$$\frac{dX(t,s)}{dt} = 2, \quad X(t_\Gamma(s), s) = x_\Gamma(s).$$

This gives $X(t,s) = x_\Gamma(s) + 2(t - t_\Gamma(s))$. Upon setting $\phi(t,s) = u(X(t,s),t)$, we observe that $\partial_t \phi(t,s) + 3\phi(t,s) = 0$, which means

$$\phi(t,s) = ce^{-3t}.$$

The initial condition implies $\phi(t_\Gamma(s), s) = u(x_\Gamma(s), t_\Gamma(s)) = e^{-t_\Gamma(s) - x_\Gamma(s)} = ce^{-3t_\Gamma(s)}$; as a result $c = e^{2t_\Gamma(s) - x_\Gamma(s)}$ and

$$\phi(t,s) = e^{2t_\Gamma(s) - x_\Gamma(s) - 3t}.$$

The implicit representation of the solution is

$$u(X(t,s),t) = e^{2t_\Gamma(s) - x_\Gamma(s) - 3t}, \qquad X(t,s) = x_\Gamma(s) + 2(t - t_\Gamma(s)).$$

Now we give the explicit representation.
We observe the following:

$$2t_\Gamma(s) - x_\Gamma(s) = 2t - X(t,s),$$

which gives

$$u(X(t,s),t) = e^{2t - X(t,s) - 3t} = e^{-X(t,s)-t}.$$

In conclusion, the explicit representation of the solution to the problem is the following:

$$u(x,t) = e^{-x-t}.$$

Question 89: Let $\Omega = \{(t,x) \in \mathbb{R}^2 : \ t > 0, \ x \geq -t\}$. Let Γ be defined by the following parameterization $\Gamma = \{x = x_\Gamma(s), t = t_\Gamma(s), s \in \mathbb{R}\}$, with $x_\Gamma(s) = s$ and $t_\Gamma(s) = -s$ if $s \leq 0$, $x_\Gamma(s) = s$ and $t_\Gamma(s) = 0$ if $s \geq 0$. Solve the following PDE (give the implicit and explicit representations):

$$u_t + 2u_x + u = 0, \quad \text{in } \Omega, \qquad u(x,t) = u_\Gamma(x,t) := \begin{cases} 1 & \text{if } x > 0 \\ 2 & \text{if } x < 0 \end{cases} \qquad \text{for all } (x,t) \text{ in } \Gamma.$$

Solution: We define the characteristics by

$$\frac{dx(t,s)}{dt} = 2, \quad x(t_\Gamma(s), s) = x_\Gamma(s).$$

This gives $x(t,s) = x_\Gamma(s) + 2(t - t_\Gamma(s))$. Upon setting $\phi(t,s) = u(x(t,s),t)$, we observe that $\partial_t \phi(t,s) + \phi(t,s) = 0$, which means

$$\phi(t,s) = ce^{-t}.$$

The initial condition implies $\phi(t_\Gamma(s), s) = u_\Gamma(x_\gamma(s), t_\Gamma(s))$; as a result $c = u_\Gamma(x_\Gamma(s), t_\Gamma(s))e^{t_\Gamma(s)}$.

$$\phi(t,s) = u_\Gamma(x_\Gamma(s), t_\Gamma(s))e^{t_\Gamma(s)-t}.$$

The implicit representation of the solution is

$$u(x(t,s),t) = u_\Gamma(x_\Gamma(s), t_\Gamma(s))e^{t_\Gamma(s)-t}, \qquad x(t,s) = x_\Gamma(s) + 2(t - t_\Gamma(s)).$$

Now we give the explicit representation.
Case 1: If $s \leq 0$, $x_\Gamma(s) = s$, $t_\Gamma(s) = -s$, and $u_\Gamma(x_\Gamma(s), t_\Gamma(s)) = 2$. This means $x(t,s) = s + 2(t+s)$ and we obtain $s = \frac{1}{3}(x - 2t)$, which means

$$u(x,t) = 2e^{-\frac{1}{3}(x-2t)-t}, \qquad \text{if } x - 2t < 0.$$

Case 2: If $s \geq 0$, $x_\Gamma(s) = s$, $t_\Gamma(s) = 0$, and $u_\Gamma(x_\Gamma(s), t_\Gamma(s)) = 1$. This means $x(t,s) = s + 2t$ and we obtain $s = x - 2t$, which means

$$u(x,t) = e^{-t}, \qquad \text{if } x - 2t > 0.$$

Question 90: Let $\Omega = \{(x,t) \in \mathbb{R}^2 \mid t > 0, x \geq \frac{1}{t}\}$. Solve the following PDE in explicit form with the method of characteristics: (Solution: $u(x,t) = (2 + \cos(s))e^{\frac{1}{s}-t}$ with $s = \frac{1}{2}[(x-2t) + \sqrt{(x-2t)^2 + 8}])$

$$\partial_t u(x,t) + 2\partial_x u(x,t) = -u(x,t), \quad \text{in } \Omega, \quad \text{and} \quad u(x,t) = 2 + \cos(x), \text{ if } x = 1/t.$$

Solution: (i) First we parameterize the boundary of Ω by setting $\Gamma = \{x = x_\Gamma(s), t = t_\Gamma(s); \ s \in \mathbb{R}\}$ with $x_\Gamma(s) = s$ and $t_\Gamma(s) = \frac{1}{s}$. This choice implies

$$u(x_\Gamma(s), t_\Gamma(s)) := u_\Gamma(s) := 2 + \cos(s).$$

(ii) We compute the characteristics

$$\partial_t X(t,s) = 2, \quad X(t_\Gamma(s),s) = x_\Gamma(s).$$

The solution is $X(t,s) = 2(t - t_\Gamma(s)) + x_\Gamma(s)$.

(iii) Set $\Phi(t,s) := u(X(t,s),t)$ and compute $\partial_t \Phi(t,s)$. This gives

$$\begin{aligned}\partial_t \Phi(t,s) &= \partial_t u(X(t,s),t) + \partial_x u(X(t,s),t)\partial_t X(t,s) \\ &= \partial_t u(X(t,s),t) + 2\partial_x u(X(t,s),t) = u(X(t,s),t) = -\Phi(t,s).\end{aligned}$$

The solution is $\Phi(t,s) = \Phi(t_\Gamma(s),s)e^{-t+t_\Gamma(s)}$.

(iv) The implicit representation of the solution is

$$X(t,s) = 2(t - t_\Gamma(s)) + x_\Gamma(s), \qquad u(X(t,s)) = u_\Gamma(s)e^{-t+t_\Gamma(s)}.$$

(v) The explicit representation is obtained by using the definitions of $-t_\Gamma(s)$, $x_\Gamma(s)$ and $u_\Gamma(s)$.

$$X(s,t) = 2(t - \frac{1}{s}) + s = 2t - \frac{2}{s} + s$$

which gives the equation

$$s^2 - s(X - 2t) - 2 = 0$$

The solutions are $s_\pm = \frac{1}{2}\left((X-2t) \pm \sqrt{(X-2t)^2 + 8}\right)$. The only legitimate solution is the positive one:

$$s = \frac{1}{2}\left((X-2t) + \sqrt{(X-2t)^2 + 8}\right)$$

The solution is

$$\begin{aligned}&u(x,t) = (2 + \cos(s))e^{\frac{1}{s} - t} \\ &\text{with } s = \frac{1}{2}((x-2t) + \sqrt{(x-2t)^2 + 8})\end{aligned}$$

7 Conservation equations

The implicit representation of the solution to the equation $\partial_t v + \partial_x q(v) = 0$, $v(x,0) = v_0(x)$, is

$$X(s,t) = q'(v_0(s))t + s; \quad v(X(s,t),t) = v_0(s). \tag{12}$$

Question 91: Consider the following conservation equation

$$\partial_t \rho + \partial_x(q(\rho)) = 0, \quad x \in (-\infty, +\infty),\ t > 0, \qquad \rho(x,0) = \rho_0(x) := \begin{cases} \frac{1}{6} & \text{if } x < 0, \\ \frac{1}{3} & \text{if } x > 0, \end{cases}$$

where $q(\rho) = \rho(2 - 3\rho)$ (and $\rho(x,t)$ is the conserved quantity). Solve this problem using the method of characteristics. Do we have a shock or an expansion wave here?

Solution: The characteristics are defined by

$$\frac{dX(s,t)}{dt} = q'(\rho) = 2(1 - 3\rho(X(s,t),t)), \quad X(s,t) = s.$$

Set $\phi(s,t) = \rho(X(s,t),t)$, then we obtain that ϕ is constant, i.e., ρ is constant along the characteristics: $\rho(X(s,t),t) = \rho(s,0) = \rho_0(s)$. As a result we can integrate the equation defining the characteristics and we obtain $X(t) = 2(1 - 3\rho_0(s))t + s$. We then have two cases depending whether s is positive or negative.

1. $s < 0$, then $\rho_0(s) = \frac{1}{6}$ and $X(s,t) = t + s$. This means
$$\rho(x,t) = \frac{1}{6} \quad \text{if} \quad x < t.$$
2. $s > 0$, then $\rho_0(s) = \frac{1}{3}$ and $X(s,t) = s$. This means
$$\rho(x,t) = \frac{1}{3} \quad \text{if} \quad x > 0.$$

We see that the characteristics cross in the region $\{t > x > 0\}$. This implies that there is a shock. The Rankin-Hugoniot relation gives the speed of this shock:

$$s = \frac{q^+ - q^-}{\rho^+ - \rho^-} = \frac{\frac{1}{6}\frac{3}{2} - \frac{1}{3}}{\frac{1}{6} - \frac{1}{3}} = \frac{1}{12}6 = \frac{1}{2}.$$

In conclusion

$$\rho = \frac{1}{6}, \quad x < \frac{t}{2},$$
$$\rho = \frac{1}{3}, \quad x > \frac{t}{2}.$$

Question 92: Consider the following conservation equation

$$\partial_t \rho + \partial_x(q(\rho)) = 0, \quad x \in (-\infty, +\infty),\ t > 0, \qquad \rho(x,0) = \rho_0(x) := \begin{cases} 3 & \text{if } x < 0, \\ 1 & \text{if } x > 0, \end{cases}$$

where $q(\rho) = \rho(2 + \rho)$ (and $\rho(x,t)$ is the conserved quantity). Solve this problem using the method of characteristics. Do we have a shock or an expansion wave here?

Solution: The characteristics are defined by

$$\frac{dX(t)}{dt} = q'(\rho) = 2(1 + \rho(x(t),t)), \quad X(0) = X_0.$$

Set $\phi(t) = \rho(X(t),t)$, then we obtain that ϕ is constant, i.e., ρ is constant along the characteristics: $\rho(X(t),t) = \rho(X_0,0) = \rho_0(X_0)$. As a result we can integrate the equation defining the characteristics and we obtain $X(t) = 2(1 + \rho_0(X_0))t + X_0$. We then have two cases depending whether X_0 is positive or negative.

1. $X_0 < 0$, then $\rho_0(X_0) = 3$ and $X(t) = 2(1+3)t + X_0 = 8t + X_0$. This means

$$\rho(x,t) = 3 \quad \text{if} \quad x < 8t.$$

2. $X_0 > 0$, then $\rho_0(X_0) = 1$ and $X(t) = 2(1+1)t + X_0 = 4t + X_0$. This means

$$\rho(x,t) = 1 \quad \text{if} \quad x > 4t.$$

We see that the characteristics cross in the region $\{8t > x > 4t\}$. This implies that there is a shock. The Rankin-Hugoniot relation gives the speed of this shock:

$$\frac{dx_s(t)}{dt} = \frac{q^+ - q^-}{\rho^+ - \rho^-} = \frac{15-3}{3-1} = 6, \qquad x_s(0) = 0.$$

In conclusion, $x_s(t) = 6t$ and

$$\begin{aligned} \rho = 3, &\quad x < x_s(t) = 6t, \\ \rho = 1, &\quad x > x_s(t) = 6t. \end{aligned}$$

Question 93: Solve the conservation equation $\partial_t \rho + \partial_x q(\rho) = 0$, $x \in (\infty, +\infty)$, $t > 0$ with flux $q(\rho) = \rho^2 + \rho$, and with the initial condition $\rho(x,0) = -1$, if $x < 0$, $\rho(x,0) = 1$, if $x > 0$. Do we have a shock or an expansion wave here?

Solution: The solution is given by the implicit representation

$$\rho(X(s,t),t) = \rho_0(s), \quad X(s,t) = s + (2\rho_0(s) + 1)t.$$

Case 1: $s < 0$. Then $\rho_0(s) = -1$ and $X(s,t) = s + (-2+1)t$. This means $s = X + t$. The solution is

$$\rho(x,t) = -1, \quad \text{if } x < t.$$

Case 2: $s < 0$. Then $\rho_0(s) = 1$ and $X(s,t) = s + (2+1)t$. This means $s = X - 3t$. The solution is

$$\rho(x,t) = 1, \quad \text{if } 3t < x.$$

We have a expansion wave. We need to consider the case $\rho_0 \in [-1,1]$ at $s = 0$.
Case 3: $s = 0$ and $\rho_0 \in [-1,1]$. Then $X(s,t) = s + (2\rho_0 + 1)t = (2\rho_0 + 1)t$. This means $\rho_0 = (X/t - 1)2$. In conclusion

$$\rho(x,t) = \frac{1}{2}\left(\frac{x}{t} - 1\right), \quad \text{if } -t < x < 3t.$$

Question 94: Solve the conservation equation $\partial_t \rho + \partial_x q(\rho) = 0$, $x \in (\infty, +\infty)$, $t > 0$ with flux $q(\rho) = \rho^4 + 2\rho$, and with the initial condition $\rho(x,0) = 1$, if $x < 0$, $\rho(x,0) = -1$, if $x > 0$. Do we have a shock or an expansion wave here?

Solution: The solution is given by the implicit representation

$$\rho(X(s,t),t) = \rho_0(s), \quad X(s,t) = s + (4\rho_0(s)^3 + 2)t.$$

We then have two cases depending whether s is positive or negative.
Case 1: $s < 0$, then $\rho_0(s) = 1$ and $X(s,t) = (4+2)t + s = 6t + s$. This means

$$\rho(x,t) = 1 \quad \text{if} \quad x < 6t.$$

Case 2: $s > 0$, then $\rho_0(s) = -1$ and $X(s,t) = (-4+2)t + s = -2t + s$. This means

$$\rho(x,t) = -1 \quad \text{if} \quad x > -2t.$$

We see that the characteristics cross in the region $\{6t > x > -2t\}$. This implies that there is a shock. The Rankin-Hugoniot relation gives the speed of this shock with $\rho^- = 1$ and $\rho^+ = -1$:

$$\frac{dx_s(t)}{dt} = \frac{q^+ - q^-}{\rho^+ - \rho^-} = \frac{-1-3}{-1-1} = 2, \qquad x_s(0) = 0.$$

In conclusion the location of the shock is $x_s(t) = 2t$ and the solution is as follows:

$$\rho = 1, \quad x < x_s(t) = 2t,$$
$$\rho = -1, \quad x > x_s(t) = 2t.$$

Question 95: Consider the following conservation equation

$$\partial_t \rho + \partial_x(q(\rho)) = 0, \quad x \in (-\infty, +\infty), \ t > 0, \qquad \rho(x,0) = \rho_0(x) := \begin{cases} \frac{1}{2} & \text{if } x < 0, \\ 1 & \text{if } x > 0, \end{cases}$$

where $q(\rho) = \rho(2-\rho)$ (and $\rho(x,t)$ is the conserved quantity). Solve this problem using the method of characteristics. Do we have a shock or an expansion wave here?

Solution: The characteristics are defined by

$$\frac{dX(t)}{dt} = q'(\rho) = 2(1 - \rho(x(t), t)), \quad X(0) = X_0.$$

Set $\phi(t) = \rho(X(t), t)$, then we obtain that ϕ is constant, i.e., ρ is constant along the characteristics: $\rho(X(t), t) = \rho(X_0, 0) = \rho_0(X_0)$. As a result we can integrate the equation defining the characteristics and we obtain $X(t) = 2(1 - \rho_0(X_0))t + X_0$. We then have two cases depending whether X_0 is positive or negative.

1. $X_0 < 0$, then $\rho_0(X_0) = \frac{1}{2}$ and $X(t) = t + X_0$. This means

$$\rho(x,t) = \frac{1}{2} \quad \text{if} \quad x < t.$$

2. $X_0 > 0$, then $\rho_0(X_0) = 1$ and $X(t) = X_0$. This means

$$\rho(x,t) = 1 \quad \text{if} \quad x > 0.$$

We see that the characteristics cross in the region $\{t > x > 0\}$. This implies that there is a shock. The Rankin-Hugoniot relation gives the speed of this shock:

$$s = \frac{q^+ - q^-}{\rho^+ - \rho^-} = \frac{\frac{3}{4} - 1}{\frac{1}{2} - 1} = \frac{1}{2}.$$

In conclusion

$$\rho = \frac{1}{2}, \quad x < \frac{t}{2},$$
$$\rho = 1, \quad x > \frac{t}{2}.$$

Question 96: Consider the following conservation equation

$$\partial_t \rho + \partial_x(q(\rho)) = 0, \quad x \in (-\infty, +\infty), \ t > 0, \qquad \rho(x,0) = \rho_0(x) := \begin{cases} 2 & \text{if } x < 0, \\ 1 & \text{if } x > 0, \end{cases}$$

where $q(\rho) = \rho(2-\rho)$ (and $\rho(x,t)$ is the conserved quantity). Solve this problem using the method of characteristics. Do we have a shock or an expansion wave here?

Solution: The characteristics are defined by

$$\frac{dX(t,x_0)}{dt} = q'(\rho) = 2(1-\rho(X(t,x_0),t)), \quad X(0,x_0) = x_0.$$

Set $\phi(t) = \rho(X(t,x_0),t)$ and insert in the equation. We obtain that $\partial_t\phi(t,x_0) = 0$; meaning that $\phi(t,x_0) = \phi(0,x_0)$, i.e., ρ is constant along the characteristics: $\rho(X(t,x_0),t) = \rho(x_0,0) = \rho_0(x_0)$. As a result we can integrate the equation defining the characteristics and we obtain $X(t,x_0) = 2(1-\rho_0(x_0))t + x_0$. The implicit representation of the solution is

$$X(t,x_0) = 2(1-\rho_0(x_0))t + x_0; \quad \rho(X(t,x_0),t) = \rho_0(x_0)$$

We then have two cases depending whether x_0 is positive or negative.
Case 1: $x_0 < 0$, then $\rho_0(x_0) = 2$ and $X(t,x_0) = 2(1-2)t + x_0 = -2t + x_0$. This means $x_0 = X(t,x_0) + 2t$ and

$$\rho(x,t) = 2 \quad \text{if} \quad x < -2t.$$

Case 2: $x_0 > 0$, then $\rho_0(x_0) = 1$ and $X(t,x_0) = 2(1-1)t + x_0 = x_0$. This means $x_0 = X(t,x_0)$ and

$$\rho(x,t) = 1 \quad \text{if} \quad 0 < x.$$

We see that there is a gap in the region $\{-2t < x < 0\}$. This implies that there is an expansion wave. We have to consider a third case $x_0 = 0$ and $\rho_0 \in (1,2)$.
Case 3: $x_0 = 0$, then $X(t,x_0) = 2(1-\rho_0)t$, i.e., $\rho_0 = 1 - \frac{X(t,x_0)}{2t}$. This means that

$$\rho(x,t) = 1 - \frac{x}{2t}, \quad \text{if} \quad -2t < x < 0.$$

Question 97: Assume $u_1 > u_2 \geq u_3 \geq 0$ and consider the following conservation equation

$$\partial_t u + u\partial_x u = 0, \quad x \in (-\infty, +\infty),\ t > 0, \qquad u(x,0) = u_0(x) := \begin{cases} 0 & \text{if } x \leq 0, \\ u_1 x & \text{if } 0 < x \leq 1, \\ u_1 & \text{if } 1 < x \leq 2, \\ u_2 & \text{if } 2 < x \leq 3, \\ u_3 & \text{if } 3 \leq x. \end{cases}$$

(i) Assume $u_2 = u_3$. Solve until the expansion catches up the shock. When does it happen?

Solution: The characteristics are defined by

$$\frac{dX(t,s)}{dt} = u(X(t,s),t), \qquad X(0,s) = s.$$

From class we know that $u(X(t,s),t)$ does not depend on time, that is to say

$$X(t,s) = u(X(0,s),0)t + s = u(s,0)t + x_0 = u_0(s)t + s.$$

Case 1: If $s \leq 0$, we have $u_0(s) = 0$ and $X(t,s) = s$; as a result, $s = X(t,s)$, and

$$u(x,t) = 0, \qquad \text{if } x \leq 0.$$

Case 2: If $0 < s \leq 1$, we have $u_0(s) = u_1 s$ and $X(t,x_0) = u_1 st + s$; as a result $s = X/(1+u_1t)$, and

$$u(x,t) = u_1 x/(1+u_1 t), \qquad \text{if } 0 < x \leq 1 + u_1 t.$$

case 3: If $1 < s \leq 2$, we have $u_0(s) = u_1$ and $X(t,x_0) = u_1 t + s$; as a result $s = X(t,s) - u_1 t$, which implies

$$u(x,t) = u_1, \qquad \text{if } 1 + u_1 t < x \leq 2 + u_1 t.$$

Case 4: If $2 < s$, we have $u_0(s) = 0$ and $X(t,s) = u_2 t + s$; as a result $s = X(t,s)$, which implies

$$u(x,t) = 0 \qquad \text{if } 2 < x.$$

We have a shock at $x = 2$ and $t = 0$. The speed of the shock is given by the Rankin-Hugoniot formula

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \frac{\frac{1}{2}u_1^2 - \frac{1}{2}u_2^2}{u_1 - u_2} = \frac{1}{2}(u_1 + u_2).$$

As a result $x_1(t) = 2 + \frac{1}{2}(u_1 + u_2)t$. This implies that the solution is

$$u(x,t) = \begin{cases} 0, & \text{if } x \leq 0, \\ u_1 x/(1+u_1 t), & \text{if } 0 < x \leq 1 + u_1 t, \\ u_1, & \text{if } 1 + u_1 t < x \leq 2 + \frac{1}{2}(u_1+u_2)t, \\ u_2, & \text{if } 2 + \frac{1}{2}(u_1+u_2)t < x. \end{cases}$$

The time T when the expansion wave catches up the shock is defined by

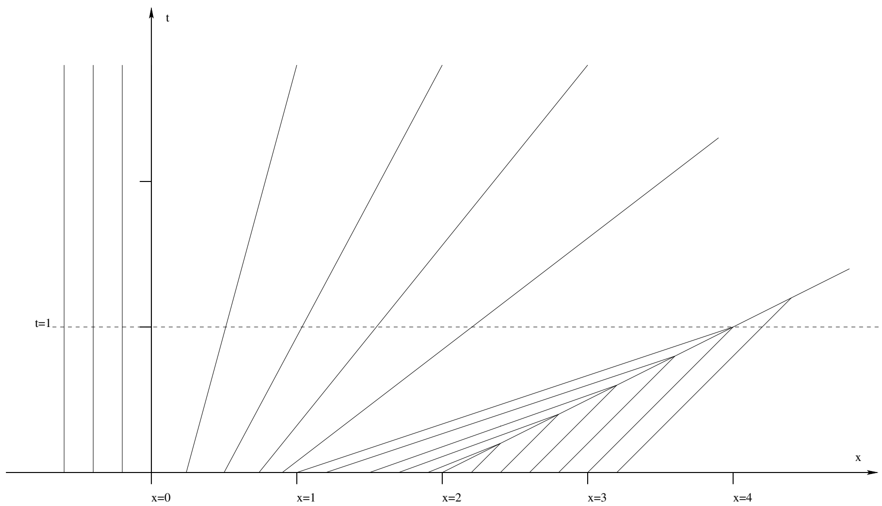
$$2 + \frac{1}{2}(u_1 + u_2)T = 1 + u_1 T,$$

that is to say

$$T = \frac{2}{u_1 - u_2}.$$

(ii) Draw the characteristics corresponding to the situation (i) with $u_1 = 2$ and $u_2 = 1$.

Solution:



(iii) Assume now that $u_1 > u_2 > u_3 = 0$. When does the first shock catches the second one?

Solution: The speed of the first shock (starting at $x = 2$ when $t = 0$) is given by the Rankin-Hugoniot formula

$$\frac{dx_1}{dt} = \frac{\frac{1}{2}u_1^2 - \frac{1}{2}u_2^2}{u_1 - u_2} = \frac{1}{2}(u_1 + u_2).$$

As a result $x_1(t) = 2 + \frac{1}{2}(u_1 + u_2)t$. The speed of the second shock (starting at $x = 3$ when $t = 0$) is given by the Rankin-Hugoniot formula

$$\frac{dx_2}{dt} = \frac{\frac{1}{2}u_2^2}{u_2} = \frac{1}{2}u_2.$$

As a result $x_2(t) = 3 + \frac{1}{2}u_2 t$.

The time T' when the two shocks are at the same location is such that $x_1(T') = x_2(T')$; that is to say,

$$2 + \frac{1}{2}(u_1 + u_2)T' = 3 + \frac{1}{2}u_2 T',$$

which gives

$$T' = \frac{2}{u_1}.$$

Note that $T > T'$ for all $u_2 > 0$. This means that the first shock catches up the second one before the fans catches the first shock.

Question 98: Consider the following conservation equation

$$\partial_t u + u\partial_x u = 0, \quad x \in (-\infty, +\infty),\ t > 0, \qquad u(x,0) = u_0(x) := \begin{cases} 0 & \text{if } x \le 0, \\ x & \text{if } 0 \le x \le 1, \\ 2 - x & \text{if } 1 \le x \le 2 \\ 0 & \text{if } 2 \le x \end{cases}$$

(i) Solve this problem using the method of characteristics for $0 \le t < 1$.

Solution: The characteristics are defined by

$$\frac{dX(t,x_0)}{dt} = u(X(t,x_0),t), \qquad X(0,x_0) = x_0.$$

From class we know that $u(X(t,x_0),t)$ does not depend on time, that is to say

$$X(t,x_0) = u(X(0,x_0),0)t + x_0 = u(x_0,0)t + x_0 = u_0(x_0)t + x_0.$$

Case 1: If $x_0 \le 0$, we have $u_0(x_0) = 0$ and $X(t, x_0) = x_0$; as a result, $x_0 = X(t, x_0)$, and

$$u(x,t) = 0, \qquad \text{if } x \le 0.$$

Case 2: If $0 \le x_0 \le 1$, we have $u_0(x_0) = x_0$ and $X(t, x_0) = tx_0 + x_0$; as a result $x_0 = X/(1+t)$, and

$$u(x,t) = x/(1+t), \qquad \text{if } 0 \le x \le 1+t.$$

case 3: If $1 \le x_0 \le 2$, we have $u_0(x_0) = 2 - x_0$ and $X(t, x_0) = t(2 - x_0) + x_0$; as a result $x_0 = (X(t, x_0) - 2t)/(1-t)$, which implies

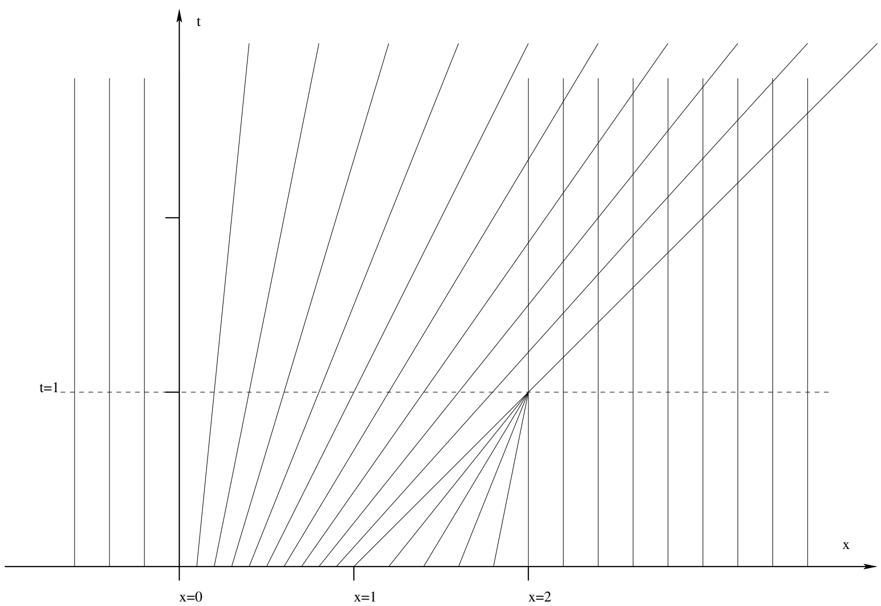
$$u(x,t) = 2 - (x-2t)/(1-t) = (2-x)/(1-t), \qquad \text{if } 1+t \le x \le 2.$$

Case 4: If $2 \le x_0$, we have $u_0(x_0) = 0$ and $X(t, x_0) = x_0$; as a result $x_0 = X(t, x_0)$, which implies

$$u(x,t) = 0 \qquad \text{if } 2 \le x.$$

(ii) Draw the characteristics for all $t > 0$ and all $x \in \mathbb{R}$.

Solution:



(iii) There is a shock forming at $t = 1$ and $x = 2$. Let $x_s(t)$ be the location of the shock as a function of t. Compute $x_s(t)$ for $t > 1$.

Solution: Let $u^-(t)$ be the value of u at the left of the shock. Conservation of mass implies

$$\frac{1}{2}u^-(t)x_s(t) = \int_{-\infty}^{+\infty} u_0(x)dx = 1.$$

The Rankin-Hugoniot formula gives

$$\dot{x}_s(t) = \frac{\frac{1}{2}(u^-(t))^2}{u^-(t)} = \frac{1}{2}u^-(t) = \frac{1}{x_s(t)}.$$

This implies

$$x_s(t)\dot{x}_s(t) = \frac{1}{2}\frac{d}{dt}(x_s(t)^2) = 1, \quad \text{with} \quad x_s(1) = 2.$$

The Fundamental Theorem of Calculus implies

$$x_s(t)^2 - 2^2 = 2(t-1),$$

which in turn implies $x_s(t) = \sqrt{2t+2}$, for all $t \geq 1$.

(iv) Write the solution for $t > 1$.

Solution: In conclusion

$$u(x,t) = \begin{cases} 0 & \text{if } x < 0, \\ \frac{x}{1+t} & \text{if } 0 \leq x < x_s(t) = \sqrt{2t+2}, \\ 0 & \text{if } \sqrt{2t+2} = x_s(t) \leq x. \end{cases}$$

Question 99: Consider the following conservation equation

$$\partial_t u + u\partial_x u = 0, \quad x \in (-\infty, +\infty),\ t > 0, \qquad u(x,0) = u_0(x) := \begin{cases} 1 & \text{if } x \leq 0, \\ 1-x & \text{if } 0 \leq x \leq 1, \\ 0 & \text{if } 1 \leq x. \end{cases}$$

(i) Solve this problem using the method of characteristics for $0 \leq t \leq 1$.

Solution: The characteristics are defined by

$$\frac{dX(t,x_0)}{dt} = u(X(t,x_0),t), \qquad X(0,x_0) = x_0.$$

From class we know that $u(X(t,x_0),t)$ does not depend on time, that is to say

$$X(t,x_0) = u(X(0,x_0),0)t + x_0 = u(x_0,0)t + x_0 = u_0(x_0)t + x_0.$$

Case 1: If $x_0 \leq 0$, we have $u_0(x_0) = 1$ and $X(t,x_0) = t + x_0$; as a result, $x_0 = X - t$, and

$$u(x,t) = 1, \qquad \text{if } x \leq t.$$

Case 2: If $0 \leq x_0 \leq 1$, we have $u_0(x_0) = 1 - x_0$ and $X(t,x_0) = t(1-x_0) + x_0$; as a result $x_0 = (X-t)/(1-t)$, and

$$u(x,t) = 1 - (t-x)/(t-1), \qquad \text{if } 0 \leq x - t \leq 1 - t,$$

which can also be re-written

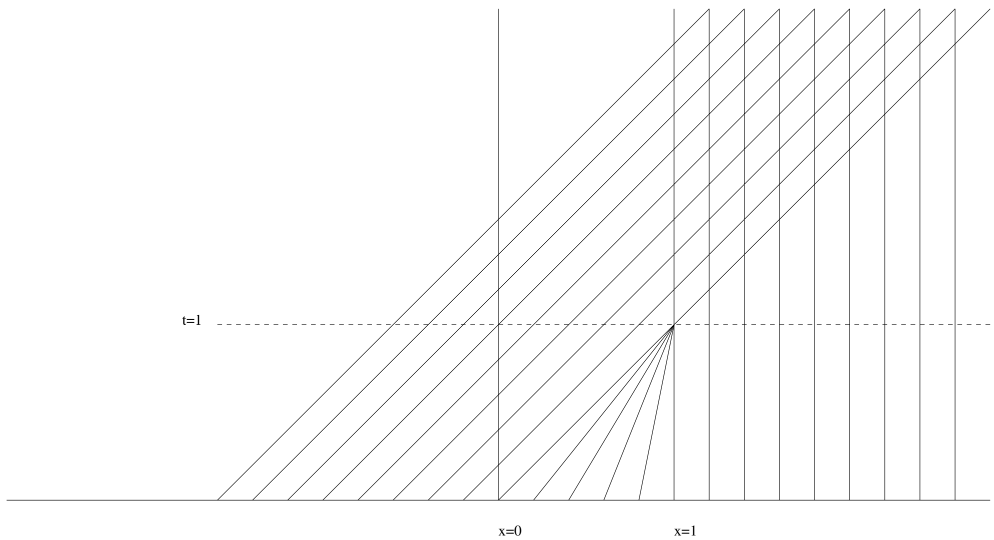
$$u(x,t) = \frac{x-1}{t-1}, \qquad \text{if } t \leq x \leq 1.$$

case 3: If $1 \leq x_0$, we have $u_0(x_0) = 0$ and $X(t,x_0) = x_0$; as a result

$$u(x,t) = 0, \qquad \text{if } 1 \leq x.$$

(ii) Draw the characteristics for all $t > 0$ and all $x \in \mathbb{R}$.

Solution:



(iii) At $t = 1$ we have $u(x, 1) = 1$ if $x < 1$ and $u(x, 1) = 0$ if $x > 1$. Solve the problem for $t > 1$.

Solution: Denote by $u_1(x)$ the solution at $t = 1$. The characteristics are $X(t, x_0) = u_1(x_0)(t - 1) + x_0$.
Case 1: If $x_0 < 1$, $u_1(x_0) = 1$ and $X(t, x_0) = t - 1 + x_0$; as a result,

$$u(x, t) = 1, \qquad \text{If } x < t.$$

Case 2: If $1 < x_0$, $u_1(x_0) = 0$ and $X(t, x_0) = x_0$; as a result,

$$u(x, t) = 0, \qquad \text{If } 1 < x.$$

The characteristics cross in the domain $\{1 < x < t\}$; as a result we have a shock. The speed of the shock is given by the Rankin-Hugoniot relation (recall that $q(u) = u^2/2$):

$$\frac{dx_s(t)}{dt} = \frac{q^+ - q^-}{u^+ - u^-} = \frac{1/2 - 0}{1 - 0} = \frac{1}{2}, \qquad x_s(1) = 1,$$

Which gives $x_s(t) = \frac{1}{2}(t + 1)$. In conclusion,

$$u(x, t) = \begin{cases} 1 & \text{If } t > 1 \text{ and } x < \frac{1}{2}(t + 1), \\ 0 & \text{If } t > 1 \text{ and } \frac{1}{2}(t + 1) < x. \end{cases}$$

Question 100: Give an explicit solution to the equation $\partial_t u + \partial_x(u^4) = 0$, where $x \in (-\infty, +\infty)$, $t > 0$, with initial data $u_0(x) = 0$ if $x < 0$, $u_0(x) = x^{\frac{1}{3}}$ if $0 < x < 1$, and $u_0(x) = 0$ if $1 < x$.

Solution: The implicit representation of the solution is

$$u(X(s, t), t) = u_0(s), \quad X(s, t) = s + 4u_0(s)^3 t.$$

Case 1: $s < 0$, then $u_0(s) = 0$ and $X(s, t) = s$. This means

$$u(x, t) = 0 \quad \text{if } x < 0.$$

Case 2: $0 < s < 1$, then $u_0(s) = s^{\frac{1}{3}}$ and $X(s, t) = s + 4st$. This means $s = X/(1 + 4t)$

$$u(x, t) = \left(\frac{x}{1 + 4t}\right)^{\frac{1}{3}} \quad \text{if } 0 < x < 1 + 4t.$$

Case 3: $1 < s$, then $u_0(s) = 0$ and $X(s,t) = s$. This means

$$u(x,t) = 0 \quad \text{if } 1 < x.$$

There is a shock starting at $x = 1$ (this is visible when one draws the characteristics).

Solution 1: The speed of the shock is given by the Rankin-Hugoniot formula

$$\frac{\mathrm{d}x_s(t)}{\mathrm{d}t} = \frac{u_+^4 - u_-^4}{u_+ - u_-}, \qquad \text{and } x_s(0) = 1,$$

where $u_+(t) = 0$ and $u_-(t) = \left(\frac{x_s(t)}{1+4t}\right)^{\frac{1}{3}}$. This gives

$$\frac{\mathrm{d}x_s(t)}{\mathrm{d}t} = u_-(t)^3 = \frac{x_s(t)}{1+4t},$$

which we re-write as follows:

$$\frac{\mathrm{d}\log(x_s(t))}{\mathrm{d}t} = \frac{1}{1+4t} = \frac{1}{4}\frac{\mathrm{d}\log(1+4t)}{\mathrm{d}t}.$$

Applying the fundamental of calculus between 0 and t gives

$$\log(x_s(t)) - \log(1) = \frac{1}{4}(\log(1+4t) - \log(1)).$$

This give

$$x_s(t) = (1+4t)^{\frac{1}{4}}.$$

Solution 2: Another (equivalent) way of solving this problem, that does not require to solve the Rankin-Hugoniot relation, consists of writing that the value of u_- is such that the total mass is conserved:

$$\int_0^{x_s(t)} u(x,t)\mathrm{d}x = \int_0^{x_s(0)} u_0(x)\mathrm{d}x = \int_0^1 x^{\frac{1}{3}}\mathrm{d}x = \frac{3}{4}$$

i.e., using the fact that $u(x,t) = (x/(1+4t))^{\frac{1}{3}}$ for all $0 \le x \le x_s(t)$, we have

$$\frac{3}{4} = (1+4t)^{-\frac{1}{3}} \int_0^{x_s(t)} x^{\frac{1}{3}}\mathrm{d}x = (1+4t)^{-\frac{1}{3}}\frac{3}{4}x_s(t)^{\frac{4}{3}}.$$

This again gives

$$x_s(t) = (1+4t)^{\frac{1}{4}}.$$

Conclusion: The solution is finally expressed as follows:

$$u(x,t) = \begin{cases} 0 & \text{if } x < 0 \\ \left(\frac{x}{1+4t}\right)^{\frac{1}{3}} & \text{if } 0 < x < (1+4t)^{\frac{1}{4}} \\ 0 & \text{if } (1+4t)^{\frac{1}{4}} < x \end{cases}$$

8 Green's function

Question 101: Let Ω be a three-dimensional domain and consider the PDE

$$\nabla^2 u = f(x), \quad x \in \Omega, \quad \text{with} \quad u(x) = h(x) \quad \text{on the boundary of } \Omega, \text{ say } \Gamma.$$

Let $G(x, x_0)$ be the Green's function of this problem (the exact expression of G does not matter; just assume that G is known). Give a representation[1] of $u(x)$ in terms of G, f and h.

Solution: By definition

$$\nabla_x^2 G(x, x_0) = \delta(x - x_0), \quad x \in \Omega, \quad \text{with} \quad G(x, x_0) = 0 \quad x \in \Gamma.$$

Then using the integration by parts formula, we obtain

$$\int_\Omega u(x)\nabla_x^2(G(x, x_0))dx = \int_\Omega \nabla_x^2(u(x))G(x, x_0)dx + \int_\Gamma u(x)\partial_n(G(x, x_0))dx - \int_\Gamma \partial_n(u(x))G(x, x_0)dx.$$

which can also be rewritten

$$u(x_0) = \int_\Omega f(x)G(x, x_0)dx + \int_\Gamma h(x)\partial_n(G(x, x_0))dx.$$

Question 102: Let f be a smooth function in $[0, 1]$. Consider the PDE

$$u - \partial_{xx} u = f(x), \quad x \in (0, 1), \qquad \partial_x u(1) + u(1) = 2, \ -\partial_x u(0) + u(0) = 1.$$

What PDE and which boundary conditions must satisfy the Green function, $G(x, x_0)$, (DO NOT compute the Green function)? Give the integral representation of u assuming $G(x, x_0)$ is known. Fully justify your answer.

Solution: Multiply the equation by $G(x, x_0)$ and integrate over $(0, 1)$:

$$\begin{aligned}
\int_0^1 f(x)G(x, x_0)dx &= \int_0^1 (u(x) - \partial_{xx}u(x))G(x, x_0)dx \\
&= \int_0^1 u(x)G(x, x_0) + \partial_x u(x)\partial_x G(x, x_0)dx - \partial_x u(1)G(1, x_0) + \partial_x u(0)G(0, x_0) \\
&= \int_0^1 u(x)(G(x, x_0) - \partial_{xx}G(x, x_0))dx + u(1)\partial_x G(1, x_0) - u(0)\partial_x G(0, x_0) \\
&\quad - \partial_x u(1)G(1, x_0) + \partial_x u(0)G(0, x_0) \\
&= \int_0^1 u(x)(G(x, x_0) - \partial_{xx}G(x, x_0))dx + u(1)\partial_x G(1, x_0) - u(0)\partial_x G(0, x_0) \\
&\quad (u(1) - 2)G(1, x_0) + (u(0) - 1)G(0, x_0) \\
&= \int_0^1 u(x)(G(x, x_0) - \partial_{xx}G(x, x_0))dx \\
&\quad + u(1)(G(1, x_0) + \partial_x G(1, x_0)) + u(0)(G(0, x_0) - \partial_x G(0, x_0)) - 2G(1, x_0) - G(0, x_0)
\end{aligned}$$

If we define $G(x, x_0)$ so that

$$G(x, x_0) - \partial_{xx}G(x, x_0) = \delta(x - x_0), \qquad G(1, x_0) + \partial_x G(1, x_0) = 0, \ G(0, x_0) - \partial_x G(0, x_0) = 0,$$

then $u(x_0)$, $x_0 \in (0, 1)$, has the following representation

$$u(x_0) = \int_0^1 f(x)G(x, x_0)dx + 2G(1, x_0) + G(0, x_0), \qquad \forall x_0 \in (0, 1).$$

[1] Hint: use $\int_\Omega \psi \nabla^2(\phi) = \int_\Omega \nabla^2(\psi)\phi + \int_\Gamma \psi \partial_n(\phi) - \int_\Gamma \partial_n(\psi)\phi$

Question 103: Consider the equation $u'(x) + u = f(x)$ for $x \in (0, 1)$ with $u(0) = a$. Let $G(x, x_0)$ be the associated Green's function. (Pay attention to the number of derivatives).
(a) Give the equation and boundary condition defining G and give an integral representation of $u(x_0)$ in terms of G, f and the boundary data a. (Do not compute G.)

Solution: The Green's function is defined by

$$-G'(x, x_0) + G(x, x_0) = \delta(x - x_0), \quad G(1, x_0) = 0.$$

We multiply the equation by u and we integrate over $(0, 1)$ (in the distribution sense),

$$\int_0^1 -G'(x, x_0)u(x)\mathsf{d}x + \int_0^1 G(x, x_0)u(x)\mathsf{d}x = u(x_0).$$

We integrates by parts and we obtain,

$$u(x_0) = \int_0^1 G(x, x_0)(u'(x) + u(x))\mathsf{d}x - G(1, x_0)u(1) + G(0, x_0)u(0)$$

Then, using the fact that $u' + u = f$ and using the boundary conditions for G and u, we obtain

$$u(x_0) = \int_0^1 G(x, x_0)f(x)\mathsf{d}x + 2G(0, x_0). \quad \forall x_0 \in (0, 1).$$

(b) Compute $G(x,x_0)$.

Solution: For $x<x_0$ and $x_0>x$ we have

$$-G'(x,x_0)+G(x,x_0)=0.$$

The solution is

$$G(x,x_0)=\begin{cases}\alpha e^x & \text{for } x<x_0\\ \beta e^x & \text{for } x>x_0.\end{cases}$$

The boundary condition $G(1,x_0)=0$ implies $\beta=0$.

For every $\epsilon>0$ we have

$$\begin{aligned}1&=\int_{x_0-\epsilon}^{x_0+\epsilon}(-G'(x,x_0)+G(x,x_0))\mathrm{d}x\\ &=G(x_0-\epsilon,x_0)-G(x_0+\epsilon,x_0)+\int_{x_0-\epsilon}^{x_0+\epsilon}G(x,x_0)\mathrm{d}x\end{aligned}$$

The term $R_\epsilon=\int_{x_0-\epsilon}^{x_0+\epsilon}G(x,x_0)\mathrm{d}x$ can be bounded as follows:

$$|R_\epsilon|\le 2\epsilon\max_{x\in[0,1]}|G(x,x_0)|=2\epsilon\alpha e^{x_0}.$$

Clearly R_ϵ goes to 0 with ϵ. As a result we obtain the jump condition

$$1=G(x_0^-,x_0)-G(x_0^+,x_0)=\alpha e^{x_0}.$$

This implies

$$\alpha=e^{-x_0}.$$

Finally

$$G(x,x_0)=\begin{cases}e^{x-x_0} & \text{for } x<x_0\\ 0 & \text{for } x>x_0.\end{cases}$$

Question 104: Consider the equation $-\partial_x(x\partial_x u(x))=f(x)$ for all $x\in(1,2)$ with $u(1)=a$ and $u(2)=b$. Let $G(x,x_0)$ be the associated Green's function.
(i) Give the equation and boundary conditions satisfied by G and give the integral representation of $u(x_0)$ for all $x_0\in(1,2)$ in terms of G, f, and the boundary data. (Do not compute G in this question).

Solution: We have a second-order PDE and the operator is clearly self-adjoint. The Green's function solves the equation

$$-\partial_x(x\partial_x G(x,x_0))=\delta(x-x_0),\quad G(1,x_0)=0,\quad G(2,x_0)=0.$$

We multiply the equation by u and integrate over the domain $(1,2)$ (in the distribution sense).

$$\langle\delta(x-x_0),u\rangle=u(x_0)=-\int_1^2\partial_x(x\partial_x G(x,x_0))u(x)\mathrm{d}x.$$

We integrate by parts and we obtain,

$$\begin{aligned}u(x_0)&=\int_1^2 x\partial_x G(x,x_0)\partial_x u(x)\mathrm{d}x-[x\partial_x G(x,x_0)u(x)]_1^2\\ &=-\int_1^2 G(x,x_0)\partial_x(x\partial_x u(x))\mathrm{d}x-2\partial_x G(2,x_0)u(2)+\partial_x G(1,x_0)u(1).\end{aligned}$$

Now, using the boundary conditions and the fact that $-\partial_x(x\partial_x u(x))=f(x)$, we finally have

$$u(x_0)=\int_1^2 G(x,x_0)f(x)\mathrm{d}x-2\partial_x G(2,x_0)b+\partial_x G(1,x_0)a.$$

(ii) Compute $G(x,x_0)$ for all $x,x_0 \in (1,2)$.

Solution: For all $x \neq x_0$ we have

$$-\partial_x(x\partial_x G(x,x_0)) = 0.$$

The solution is

$$G(x,x_0) = \begin{cases} a\log(x) + b & \text{if } 1 < x < x_0 \\ c\log(x) + d & \text{if } x_0 < x < 2 \end{cases}$$

The boundary conditions give $b = 0$ and $d = -c\log(2)$; as a result,

$$G(x,x_0) = \begin{cases} a\log(x) & \text{if } 1 < x < x_0 \\ c\log(x/2) & \text{if } x_0 < x < 2 \end{cases}$$

G must be continuous at x_0,

$$a\log(x_0) = c\log(x_0) - c\log(2)$$

and must satisfy the gap condition

$$-\int_{x_0-\epsilon}^{x_0+\epsilon} \partial_x(x\partial_x G(x,x_0))\mathsf{d}x = 1, \qquad \forall \epsilon > 0.$$

This gives

$$\begin{aligned} -x_0\left(\partial_x G(x_0^+,x_0) - G(x_0^-,x_0)\right) &= 1 \\ -x_0\left(\frac{c}{x_0} - \frac{a}{x_0}\right) &= 1 \end{aligned}$$

This gives

$$a - c = 1.$$

In conclusion $\log(x_0) = -c\log 2$ and

$$c = -\log(x_0)/\log(2), \quad a = 1 - \log(x_0)/\log(2) = \log(2/x_0)/\log(2).$$

This means

$$G(x,x_0) = \begin{cases} \frac{\log(2/x_0)}{\log(2)}\log(x) & \text{if } 1 < x < x_0 \\ \frac{\log(x_0)}{\log(2)}\log(2/x) & \text{if } x_0 < x < 2 \end{cases}$$

Question 105: Consider the equation $u'(x) + u = f(x)$ for $x \in (0,1)$ with $u(0) = a$. Let $G(x,x_0)$ be the associated Green's function. (Pay attention to the number of derivatives).
(a) Give the equation and boundary condition defining G and give an integral representation of $u(x_0)$ in terms of G, f and the boundary data a. (Do not compute G.)

Solution: The Green's function is defined by

$$-G'(x,x_0) + G(x,x_0) = \delta(x-x_0), \quad G(1,x_0) = 0.$$

We multiply the equation by u and we integrate over $(0,1)$ (in the distribution sense),

$$\int_0^1 -G'(x,x_0)u(x)\mathsf{d}x + \int_0^1 G(x,x_0)u(x)\mathsf{d}x = u(x_0).$$

We integrates by parts and we obtain,

$$u(x_0) = \int_0^1 G(x,x_0)(u'(x) + u(x))\mathsf{d}x - G(1,x_0)u(1) + G(0,x_0)u(0)$$

Then, using the fact that $u' + u = f$ and using the boundary conditions for G and u, we obtain

$$u(x_0) = \int_0^1 G(x,x_0)f(x)\mathsf{d}x + aG(0,x_0). \quad \forall x_0 \in (0,1).$$

(b) Compute $G(x,x_0)$.

Solution: For $x < x_0$ and $x_0 > x$ we have

$$-G'(x,x_0) + G(x,x_0) = 0.$$

The solution is

$$G(x,x_0) = \begin{cases} \alpha e^x & \text{for } x < x_0 \\ \beta e^x & \text{for } x > x_0. \end{cases}$$

The boundary condition $G(1,x_0) = 0$ implies $\beta = 0$.

For every $\epsilon > 0$ we have

$$\begin{aligned} 1 &= \int_{x_0-\epsilon}^{x_0+\epsilon} (-G'(x,x_0) + G(x,x_0))\mathsf{d}x \\ &= G(x_0-\epsilon,x_0) - G(x_0+\epsilon,x_0) + \int_{x_0-\epsilon}^{x_0+\epsilon} G(x,x_0)\mathsf{d}x \end{aligned}$$

The term $R_\epsilon = \int_{x_0-\epsilon}^{x_0+\epsilon} G(x,x_0)\mathsf{d}x$ can be bounded as follows:

$$|R_\epsilon| \le 2\epsilon \max_{x\in[0,1]} |G(x,x_0)| = 2\epsilon\alpha e^{x_0}.$$

Clearly R_ϵ goes to 0 with ϵ. As a result we obtain the jump condition

$$1 = G(x_0^-,x_0) - G(x_0^+,x_0) = \alpha e^{x_0}.$$

This implies

$$\alpha = e^{-x_0}.$$

Finally

$$G(x,x_0) = \begin{cases} e^{x-x_0} & \text{for } x < x_0 \\ 0 & \text{for } x > x_0. \end{cases}$$

Question 106: Consider the equation $-\partial_x(x\partial_x u(x)) = f(x)$ for all $x \in (1,2)$ with $u(1) = a$ and $u(2) = b$. Let $G(x,x_0)$ be the associated Green's function.
(i) Give the equation and boundary conditions satisfied by G and give the integral representation of $u(x_0)$ for all $x_0 \in (1,2)$ in terms of G, f, and the boundary data. (Do not compute G in this question).

Solution: We have a second-order PDE and the operator is clearly self-adjoint. The Green's function solves the equation

$$-\partial_x(x\partial_x G(x,x_0)) = \delta(x-x_0), \quad G(1,x_0) = 0, \quad G(2,x_0) = 0.$$

We multiply the equation by u and integrate over the domain $(1,2)$ (in the distribution sense).

$$\langle \delta(x-x_0), u\rangle = u(x_0) = -\int_1^2 \partial_x(x\partial_x G(x,x_0))u(x)\mathsf{d}x.$$

We integrate by parts and we obtain,

$$\begin{aligned} u(x_0) &= \int_1^2 x\partial_x G(x,x_0)\partial_x u(x)\mathsf{d}x - [x\partial_x G(x,x_0)u(x)]_1^2 \\ &= -\int_1^2 G(x,x_0)\partial_x(x\partial_x u(x))\mathsf{d}x - 2\partial_x G(2,x_0)u(2) + \partial_x G(1,x_0)u(1). \end{aligned}$$

Now, using the boundary conditions and the fact that $-\partial_x(x\partial_x u(x)) = f(x)$, we finally have

$$u(x_0) = \int_1^2 G(x,x_0)f(x)\mathsf{d}x - 2\partial_x G(2,x_0)b + \partial_x G(1,x_0)a.$$

(ii) Compute $G(x,x_0)$ for all $x, x_0 \in (1,2)$.

Solution: For all $x \neq x_0$ we have

$$-\partial_x(x\partial_x G(x,x_0)) = 0.$$

The solution is

$$G(x,x_0) = \begin{cases} a\log(x)+b & \text{if } 1<x<x_0 \\ c\log(x)+d & \text{if } x_0<x<2 \end{cases}$$

The boundary conditions give $b=0$ and $d=-c\log(2)$; as a result,

$$G(x,x_0) = \begin{cases} a\log(x) & \text{if } 1<x<x_0 \\ c\log(x/2) & \text{if } x_0<x<2 \end{cases}$$

G must be continuous at x_0,

$$a\log(x_0) = c\log(x_0) - c\log(2)$$

and must satisfy the gap condition

$$-\int_{x_0-\epsilon}^{x_0+\epsilon} \partial_x(x\partial_x G(x,x_0))\mathsf{d}x = 1, \qquad \forall \epsilon > 0.$$

This gives

$$\begin{aligned} -x_0\left(\partial_x G(x_0^+,x_0) - G(x_0^-,x_0)\right) &= 1 \\ -x_0\left(\frac{c}{x_0} - \frac{a}{x_0}\right) &= 1 \end{aligned}$$

This gives

$$a - c = 1.$$

In conclusion $\log(x_0) = -c\log 2$ and

$$c = -\log(x_0)/\log(2), \quad a = 1-\log(x_0)/\log(2) = \log(2/x_0)/\log(2).$$

This means

$$G(x,x_0) = \begin{cases} \frac{\log(2/x_0)}{\log(2)}\log(x) & \text{if } 1<x<x_0 \\ \frac{\log(x_0)}{\log(2)}\log(2/x) & \text{if } x_0<x<2 \end{cases}$$

Question 107: Consider the equation $\partial_{xx}u(x) = f(x)$, $x\in(0,L)$, with $u(0)=a$ and $\partial_x u(L) = b$.

(a) Compute the Green's function of the problem.

Solution: Let x_0 be a point in $(0,L)$. The Green's function of the problem is such that

$$\partial_{xx}G(x,x_0) = \delta_{x_0}, \quad G(0,x_0)=0, \quad \partial_x G(L,x_0) = 0.$$

The following holds for all $x\in(0,x_0)$:

$$\partial_{xx}G(x,x_0) = 0.$$

This implies that $G(x,x_0) = ax+b$ in $(0,x_0)$. The boundary condition $G(0,x_0)=0$ gives $b=0$. Likewise, the following holds for all $x\in(x_0,L)$:

$$\partial_{xx}G(x,x_0) = 0.$$

This implies that $G(x,x_0) = cx + d$ in (x_0, L). The boundary condition $\partial_x G(L,x_0) = 0$ gives $c = 0$. The continuity of $G(x,x_0)$ at x_0 implies that $ax_0 = d$. The condition

$$\int_{-\epsilon}^{\epsilon} \partial_{xx} G(x,x_0)\mathsf{d}x = 1, \qquad \forall \epsilon > 0,$$

gives the so-called jump condition: $\partial_x G(x_0^+, x_0) - \partial_x G(x_0^-, x_0) = 1$. This means that $0 - a = 1$, i.e., $a = -1$ and $d = -x_0$. In conclusion

$$G(x,x_0) = \begin{cases} -x & \text{if} \le x \le x_0, \\ -x_0 & \text{otherwise.} \end{cases}$$

(b) Give the integral representation of u using the Green's function.

Solution: Let x_0 be a point in $(0,L)$. The definition of the Dirac measure at x_0 is such that

$$\begin{aligned} u(x_0) &= \langle \delta_{x_0}, u \rangle = \langle \partial_{xx} G(\cdot, x_0), u \rangle \\ &= -\int_0^L \partial_x G(x,x_0)\partial_x u(x)\mathsf{d}x + [\partial_x G(x,x_0)u(x)]_0^L \\ &= \int_0^L G(x,x_0)\partial_{xx} u(x)\mathsf{d}x - [G(x,x_0)\partial_x u(x)]_0^L + [\partial_x G(x,x_0)u(x)]_0^L \\ &= \int_0^L G(x,x_0)f(x)\mathsf{d}x - G(L,x_0)\partial_x u(L) + G(0,x_0)\partial_x u(0) + \partial_x G(L,x_0)u(L) - \partial_x G(0,x_0)u(0). \end{aligned}$$

This finally gives the following representation of the solution:

$$u(x_0) = \int_0^L G(x,x_0)f(x)\mathsf{d}x - G(L,x_0)b - \partial_x G(0,x_0)a$$

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9 Fredholm alternative

Question 108: (a) Compute the solution set of the equation

$$u'' + u = 0,\ x \in (0, 2\pi) \quad \text{with} \quad u(0) = u(2\pi),\ u'(0) = u'(2\pi).$$

Solution: clearly $u(x) = a\cos(x) + b\sin(x)$ where a and b are arbitrary numbers. The solution set is a two-dimensional vector space spanned by $\cos(x)$ and $\sin(x)$.

(b) Do the following equation have solution(s)? (Hint: think of the Fredholm alternative)

$$u''(x) + u(x) = \sin(2x),\ \forall x \in (0, 2\pi) \quad \text{with} \quad u(0) = u(2\pi),\ u'(0) = u'(2\pi).$$

Solution: We are in the second case of the Fredholm alternative, i.e., the null space of the operator is not $\{0\}$. We have to verify that $\sin(2x)$ is orthogonal to $\cos(x)$ and $\sin(x)$, which is clearly true. In conclusion the equation has solutions.

(c) Do the following equation have solution(s)? (Hint: think of the Fredholm alternative)

$$u''(x) + u(x) = \sin(x),\ \forall x \in (0, 2\pi) \quad \text{with} \quad u(0) = u(2\pi),\ u'(0) = u'(2\pi).$$

Solution: We are in the second case of the Fredholm alternative, i.e., the null space of the operator is not $\{0\}$. We have to verify that $\sin(x)$ is orthogonal to $\cos(x)$ and $\sin(x)$, which is clearly wrong. In conclusion the equation has no solution.

Question 109: (i) Use the energy method to compute the null space of the self-adjoint operator $L : \{v \in \mathcal{C}^2[0,1]; v'(0) = v'(1) = 0\} \longrightarrow \mathcal{C}^0[0,1]$ defined by $L(v) := -v''$.

Solution: Let v be a member of the null space. Then $L(v) = 0$ if and only if

$$-v''(x) = 0, \quad \forall x \in [0,1], \quad v'(0) = 0, \quad v'(1) = 0.$$

Multiply the equation by v and integrate over $[0,1]$:

$$0 = -\int_0^1 v''(x)v(x)\mathsf{d}x = \int_0^1 v'(x)v'(x)\mathsf{d}x - v'(1)v(1) + v'(0)v(0) = \int_0^1 (v'(x))^2\mathsf{d}x.$$

This implies that $v'(x) = 0$ for all $x \in [0,1]$, which in turns implies that $v(x) = a$ where a is an arbitrary constant, i.e., $\mathsf{Null}(L) \subset \mathsf{span}(1)$. Conversely it is clear that $v(x) = a$ is in the null space of L, i.e., $\mathsf{span}(1) \subset \mathsf{Null}$. In conclusion

$$\mathsf{Null}(L) = \mathsf{span}(1).$$

(ii) Apply the Fredholm alternative to deduce whether the following equation has a solution, and if it does whether it is unique: $-u''(x) = \frac{1}{2} - x$, where $x \in [0,1]$ and $u'(0) = 0$, $u'(1) = 0$.

Solution: The problem consists of finding u in $H := \{v \in \mathcal{C}^2[0,1]; v'(0) = v'(1) = 0\}$ so that $Lu = \frac{1}{2} - x$. From (i) we infer that the null space of L is not reduced to $\{0\}$, this means that we are in the second case of the Fredholm alternative. There exists a solution if and only if $\int_0^1 (\frac{1}{2} - x)v(x)\mathsf{d}x = 0$ for all v in the null space of $L^t = L$. Let v be in the null space of L. We have seen in (i) that $v(x) = a$, where $a \in \mathbb{R}$; this implies

$$\int_0^1 (\frac{1}{2} - x)v(x)\mathsf{d}x = a\int_0^1 (\frac{1}{2} - x)\mathsf{d}x = a(\frac{1}{2} - \frac{1}{2}) = 0.$$

In conclusion the condition $\int_0^1 (\frac{1}{2} - x)v(x)\mathsf{d}x = 0$ for all $v \in \mathsf{Null}(L)$ is satisfied. This means that the problem has a solution but the solution is not unique.

Question 110: Consider the equation $-u''(x) = f(x)$ for $x \in (0,1)$ with $u'(0) = 2$ and $u(1) = 1$. Let $G(x, x_0)$ be the associated Green's function. (Pay attention to the minus sign).
(a) Give an expression of $u(x)$ in terms of G, f and the boundary data.

Solution: The Green's function is defined by

$$-G''(x,x_0) = \delta(x - x_0), \quad G'(0,x_0) = 0, \quad G(1,x_0) = 0.$$

We multiply the equation by u and we integrate over $(0,1)$ (in the distribution sense),

$$\int_0^1 -G''(x,x_0)u(x)dx = u(x_0).$$

We integrates by parts twice and we obtain,

$$\begin{aligned} u(x_0) &= \int_0^1 G'(x,x_0)u'(x)dx - G'(1,x_0)u(1) + G'(0,x_0)u(0) \\ &= -\int_0^1 G(x,x_0)u''(x)\mathrm{d}x + G(1,x_0)u'(1) - G(0,x_0)u'(0) - G'(1,x_0)u(1) + G'(0,x_0)u(0). \end{aligned}$$

Then, using the boundary conditions for G and u, we obtain

$$u(x_0) = \int_0^1 G(x,x_0)f(x)dx - 2G(0,x_0) - G'(1,x_0), \quad \forall x_0 \in (0,1).$$

(b) Compute $G(x, x_0)$.

Solution: For $x < x_0$ we have

$$G(x,x_0) = ax + b.$$

The boundary condition $G'(0,x_0) = 0$ implies $a = 0$; hence, $G(x,x_0) = b$. For $x_0 < x$ we have

$$G(x,x_0) = c(x-1) + d.$$

The boundary condition $G(1,x_0) = 0$ implies $d = 0$; hence, $G(x,x_0) = c(x-1)$. Moreover we have

$$1 = -\int_0^1 G''(x,x_0)\mathrm{d}x = -G'(1,x_0) + G'(0,x_0) = -c,$$

meaning $c = -1$. The continuity of G at x_0 implies

$$b = 1 - x_0.$$

As a result,

$$G(x,x_0) = \begin{cases} 1 - x_0, & \text{if } 0 \le x \le x_0, \\ 1 - x, & \text{if } x_0 \le x \le 1. \end{cases}$$

10 Classification of PDEs

Question 111: Consider the following PDE's:

$$\begin{aligned}
\partial_y u(x,y) + 3\partial_{xx} u(x,y) &= f(x,y), \quad y>0, \quad x\in(0,L) \\
&u(x,0)=1, \quad u(0,y)=3, \quad \partial_x u(L,y)=2 \qquad (13)\\
\partial_y u(x,y) - 3\partial_{xx} u(x,y) &= f(x,y), \quad y>0, \quad x\in(0,L) \\
&u(x,0)=1, \quad u(0,y)=3, \quad \partial_x u(0,y)=2, \quad \partial_x u(L,y)=2 \qquad (14)\\
\partial_y u(x,y) - 3\partial_{xx} u(x,y) &= f(x,y), \quad y>0, \quad x\in(0,L) \\
&u(x,0)=1, \quad u(0,y)=3, \quad \partial_x u(L,y)=2 \qquad (15)\\
\partial_{yy} u(x,y) - 3\partial_{xx} u(x,y) &= f(x,y), \quad y>0, \quad x\in(0,L) \\
&u(x,0)=1, \quad u(0,y)=3, \quad \partial_x u(L,y)=2 \qquad (16)\\
\partial_{yy} u(x,y) - 3\partial_{xx} u(x,y) &= f(x,y), \quad y>0, \quad x\in(0,L) \\
&u(x,0)=1, \quad \partial_y u(x,0)=1, \quad u(0,y)=2, \quad \partial_x u(L,y)=3 \qquad (17)\\
\partial_{yy} u(x,y) + 3\partial_{xx} u(x,y) &= f(x,y), \quad y>0, \quad x\in(0,L) \\
&u(x,0)=1, \quad \partial_y u(x,0)=1, \quad u(0,y)=2, \quad \partial_x u(L,y)=3 \qquad (18)\\
-\partial_{yy} u(x,y) - \partial_{xx} u(x,y) &= f(x,y), \quad y\in(0,H), \quad x\in(0,L) \\
&u(x,0)=1, \quad u(x,H)=1, \quad u(0,y)=2, \quad \partial_x u(L,y)=3 \qquad (19)\\
-\partial_{yy} u(x,y) - \partial_{xx} u(x,y) &= f(x,y), \quad y\in(0,H), \quad x\in(0,L) \\
&u(x,0)=1, \quad u(x,H)=1, \quad \partial_x u(x,H)=2, \quad \partial_x u(L,y)=3 \qquad (20)\\
-\partial_{yy} u(x,y) - \partial_{xx} u(x,y) &= f(x,y), \quad y\in(0,H), \quad x\in(0,L) \\
&u(x,0)=1, \quad u(L,y)=2, \quad \partial_x u(0,y)=3 \qquad (21)\\
\partial_y u(x,y) + 3\partial_x u(x,y) &= f(x,y), \quad y\in(0,H), \quad x\in(0,L) \\
&u(x,0)=1, \quad u(x,H)=1, \quad u(L,y)=2, \quad \partial_x u(0,y)=3 \qquad (22)\\
\partial_y u(x,y) + 3\partial_x u(x,y) &= f(x,y), \quad y>0 \quad x\in(0,L) \\
&u(x,0)=1, \quad u(0,y)=2, \quad \partial_x u(L,y)=3 \qquad (23)\\
\partial_y u(x,y) + 3\partial_x u(x,y) &= f(x,y), \quad y>0 \quad x\in(0,L) \\
&u(x,0)=1, \quad u(0,y)=2 \qquad (24)
\end{aligned}$$

Which one is the

- Heat equation? ____________ 9
- Laplace equation? ____________ 13
- Transport equation? ____________ 18
- Wave equation? ____________ 11