ENERGY METHODS FOR THE HEAT EQUATION
MATH 602

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1. Introduction

We are interested in the qualitative behavior of solutions to the heat equation, which we take to be of the form:

$$\begin{cases} \partial_t u - k\partial_{xx} u = Q(x), & (1.1)\\ u(0,t) = 0, & (1.2)\\ u(L,t) = 0, & (1.3)\\ u(x,0) = u_0(x). & (1.4)\end{cases}$$

This can be thought of as describing heat flow in a (“one dimensional”) metal rod or wire with conduction coefficient $k > 0$ (we assume k is constant here), with the ends of the rod immersed in a cold (“zero temperature ”) bath. There is a source of heat $Q(x)$ which does not depend on time, and the initial heat distribution is given by u_0.

We want to learn something about what this equation predicts, without having to solve the equation first. This is a powerful tool: it gives us something which is true for *all* solutions. We begin with some notation.

2. Notation

Sometimes, we write $u(t)$ or $u(\cdot,t)$ to mean $u(x,t)$. This is just a short-hand notation for convenience, when we are focused on time more than on space. Let $f(x)$ and $g(x)$ be two functions on an interval (a,b). We define

$$\langle f,g\rangle \equiv (f,g) := \int_a^b f(x)g(x)\,dx \qquad \text{(the } L^2 \text{ inner product)}$$

$$\|f\| \equiv \|f\|_{L^2} := \left(\int_a^b |f(x)|^2\,dx\right)^{1/2} \equiv \sqrt{(f,f)} \qquad \text{(the } L^2 \text{ norm)}$$

$$\|f\|_{L^p} := \left(\int_a^b |f(x)|^p\,dx\right)^{1/p} \qquad \text{(the } L^p \text{ norm, } 1 \le p < \infty)$$

Fact: L^p has an inner-product only when $p = 2$. This is part of what makes L^2 so useful.
Notice the following L^2 analogies with vectors $\mathbf{u} = (u_1,u_2,u_3)$ and $\mathbf{v} = (v_1,v_2,v_3)$:

$$\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3$$

$$|\mathbf{u}| = \sqrt{\mathbf{u}\cdot\mathbf{u}} = \sqrt{u_1^2 + u_2^2 + u_3^2}$$

Historical note: The L in L^p stands for *Lesbegue*, after the French mathematician Henri Lebesgue.

3. Inequalities

Much of the work in both analytical and numerical PDEs involves inequalities. Here are two of the most useful and famous ones. Let $f(x)$ and $g(x)$ be two functions on an interval $(0,L)$. Then:

$$(f,g) \le \|f\|\|g\| \qquad \text{(Cauchy-Schwarz inequality)} \tag{3.1}$$

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Suppose that $f(0) = 0$ or $f(L) = 0$ (this can be generalized greatly). Then,

$$\|f\|_{L^2} \leq L\|f'\|_{L^2} \qquad \text{(Poincaré inequality)} \tag{3.2}$$

Also, for any numbers $a \geq 0$, $b \geq 0$ and $\epsilon > 0$, we have Young's inequality:

$$ab \leq \frac{a^2}{2} + \frac{b^2}{2} \tag{3.3}$$

and Young's ϵ-inequality

$$ab \leq \frac{a^2}{2\epsilon} + \frac{\epsilon b^2}{2} \tag{3.4}$$

We proved these in class, and the proofs are not the main focus, so we will not reproduce the proofs here. For now, let us use them.

4. Energy Methods

Energy methods generally consist of taking the inner-product of the equation with the solution u, or u'', or some function of u. Here, we will just take the inner product with u. Recall that "taking the inner product with u" just means multiplying the equation by u and integrating in space. Taking the inner product of the heat equation with u yields

$$(\partial_t u, u) - k(\partial_{xx} u, u) = (Q, u) \tag{4.1}$$

Now, we notice that

$$\begin{aligned}
(\partial_t u, u) &= \int_0^L \frac{\partial u}{\partial t} u \, dx \\
&= \int_0^L \frac{1}{2}\frac{\partial u^2}{\partial t} \, dx && \text{by the chain rule,} \\
&= \frac{1}{2}\frac{d}{dt}\int_0^L u^2 \, dx && \text{since integral is in } x\text{, not } t \\
&= \frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2
\end{aligned}$$

Also, recall the integration by parts formula in the form:

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x)\,dx$$

Using this, we notice that

$$\begin{aligned}
(\partial_{xx} u, u) &= \int_0^L \partial_x[\partial_x u(x)] \cdot u(x)\,dx \\
&= [\partial_x u(x)] \cdot u(x)\Big|_0^L - \int_0^L \partial_x u(x) \cdot \partial_x u(x)\,dx \\
&= [\partial_x u(L)] \cdot u(L) - [\partial_x u(0)] \cdot u(0) - \int_0^L (\partial_x u(x))^2\,dx \\
&= -\int_0^L (\partial_x u(x))^2\,dx && \text{since } u(0) = 0 \text{ and } u(L) = 0 \\
&= -\|\partial_x u\|_{L^2}^2
\end{aligned}$$

Thus, (4.1) the following equation, which is known as the **energy equality** or **energy balance equation** for the heat equation.

$$\boxed{\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + k\|\partial_x u\|_{L^2}^2 = (Q, u)} \tag{4.2}$$

We can get something from this already:

Suppose there is no source for the heat equation (1.1)*, that is $Q = 0$. Then the energy is decaying.* To see this, note that when $Q = 0$ the energy equation (4.2) yields:

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 = -k\|\partial_x u\|^2 \leq 0$$

since $k > 0$. Thus, the quantity $\frac{1}{2}\|u\|^2$ never increases in time when there is no source Q. This is to be expected from our usual notion of heat, but it is nice to see that our equation predicts this. We can also learn about the rate of energy decay, as follows.

The Poincaré inequality tell us that $\|u\|^2 \leq L^2\|\partial_x u\|^2$. Therefore, from (4.2),

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \frac{k}{L^2}\|u\|^2 \leq (Q, u) \tag{4.3}$$

NOTE: In class, I wrote kL^2 instead of $\frac{k}{L^2}$, which was a mistake. It is fixed here.

Next, using the Cauchy-Schwarz inequality, we have $(Q, u) \leq |(Q, u)| \leq \|Q\|\|u\|$, so that

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \frac{k}{L^2}\|u\|^2 \leq \|Q\|\|u\|$$

The next step is to use Young's ϵ-inequality with $\epsilon = k/L^2$, $a = \|Q\|$ and $b = \|u\|$.

$$(Q, u) \leq \|Q\|\|u\| \leq \frac{\|Q\|^2}{2(k/L^2)} + \frac{(k/L^2)\|u\|^2}{2} = \frac{1}{2}\frac{L^2}{k}\|Q\|^2 + \frac{1}{2}\frac{k}{L^2}\|u\|^2$$

Using this with (4.3), we can further estimate

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \frac{k}{L^2}\|u\|^2 \leq \frac{1}{2}\frac{L^2}{k}\|Q\|^2 + \frac{1}{2}\frac{k}{L^2}\|u\|^2$$

Subtracting $\frac{1}{2}\frac{k}{L^2}\|u\|^2$ from both sides and multiplying by 2 yields

$$\frac{d}{dt}\|u\|^2 + \frac{k}{L^2}\|u\|^2 \leq \frac{L^2}{k}\|Q\|^2 \tag{4.4}$$

We are almost done, but we need to isolate $\|u\|^2$. To simplify our discussion, let us set

$$y \equiv y(t) := \|u(\cdot, t)\|^2 \qquad \text{and} \qquad C := \frac{L^2}{k}\|Q\|^2.$$

Then, (4.4) becomes

$$\frac{dy}{dt} + \frac{k}{L^2}y \leq C$$

Using the trick from ODEs of multiplying by an integrating factor, we multiply the equation by e^{kt/L^2} to obtain

$$e^{kt/L^2}\frac{dy}{dt} + \frac{k}{L^2}e^{kt/L^2}y \leq Ce^{kt/L^2}$$

By the chain rule, $\frac{d}{dt}(e^{kt/L^2}y) = e^{kt/L^2}\frac{dy}{dt} + \frac{k}{L^2}e^{kt/L^2}y$, so the inequality becomes

$$\frac{d}{dt}(e^{kt/L^2}y) \leq Ce^{kt/L^2}$$

Let us now integrate both sides in time t over $[0, T]$:

$$\int_0^T \frac{d}{dt}(e^{kt/L^2}y(t))\,dt \leq C\int_0^T e^{kt/L^2}\,dt$$

Using the Fundamental Theorem of Calculus on the left-hand side, and simply computing the right-hand side, we obtain

$$e^{kT/L^2}y(T) - e^{k0/L^2}y(0) \leq \frac{CL^2}{k}(e^{kT/L^2} - e^{k0/L^2})$$

NOTE: In class, missed the extra constant in front of C. It is correct here.

That is,

$$e^{kT/L^2} y(T) \le y(0) + \frac{CL^2}{k}(e^{kT/L^2} - 1)$$

Multiplying by e^{-kT/L^2}, we find

$$y(T) \le y(0)e^{-kT/L^2} + \frac{CL^2}{k}(1 - e^{-kT/L^2})$$

Noting that $T \ge 0$ is arbitrary we can replace is by t. Replacing also y and C by their values, and noting that $y(0) = \|u_0\|^2$, we find

$$\boxed{\|u(t)\|^2 \le \|u_0\|^2 e^{-kt/L^2} + \frac{L^4}{k^2}(1 - e^{-kt/L^2})\|Q\|^2.} \tag{4.5}$$

5. Results

Now, we can use (6.5) extract more information about the solution. First of all:

- ***Suppose $Q \equiv 0$. Then the solution decays exponentially.***
  This follows immediately from (6.5), since if $Q \equiv 0$,
  $$\|u(t)\|^2 \le \|u_0\|^2 e^{-kt/L^2}.$$
- Next, what happens as $t \to \infty$? If the limit exists, we find from (6.5),
  $$\lim_{t\to\infty} \|u(t)\|^2 \le \tfrac{L^4}{k^2}\|Q\|^2.$$
  Note that the initial condition does not appear here. The heat equation tends to "forget" its initial condition.
- What happens when $t = 0$? From (6.5), we get
  $$\|u(0)\|^2 \le \|u_0\|^2$$
  This gives no information since $\|u(0)\|^2 = \|u_0\|^2$, but it is a good check. If we had found $\|u(0)\|^2 \le \frac{1}{2}\|u_0\|^2$ or something, we would have been wrong.

6. Uniqueness

We now have perhaps the most important result of the energy method: uniqueness, and continuous dependence on initial data. Suppose u is a solution to (1.1) with initial data u_0, and v is a solution to (1.1) with initial data v_0. Set $w = u - v$ and $w_0 = u_0 - v_0$. Then, subtracting the equations for u and v, the Q cancels, and we find

$$\begin{cases} \partial_t w - k\partial_{xx} w = 0, & (6.1) \\ w(0,t) = 0, & (6.2) \\ w(L,t) = 0, & (6.3) \\ w(x,0) = w_0(x). & (6.4) \end{cases}$$

This is just the same equation we solved earlier, with w instead of u, and $Q = 0$!! Therefore, exactly the same analysis holds, and we find $\|w(t)\|^2 \le \|w_0\|^2 e^{-kt/L^2}$. That is,

$$\boxed{\|u(t) - v(t)\|^2 \le \|u_0 - v_0\|^2 e^{-kt/L^2}.} \tag{6.5}$$

From this, we see that if u_0 and v_0 are very "close", that is, $\|u_0 - v_0\|$ is very small, then$\|u(t) - v(t)\|$ is very small, so $u(t)$ and $v(t)$ are very "close" for all time $t \ge 0$. Thus, ***if the measurement of the initial heat has small errors, the prediction of the heat evolution will only have small errors.***

Furthermore, if $u_0 = v_0$, then $w_0 = 0$, so $\|w(t)\|^2 \le \|w_0\|^2 e^{-kt/L^2} = 0$, so $u(t) - v(t) = w(t) = 0$. Thus, if the initial conditions are the same, the solutions are the same. That is:

Solutions to the heat equation with a given initial condition, given boundary conditions, and a given source Q, are unique.