LAST NAME: Key FIRST NAME: ______________ Exam 1

MATH 602, Differential Equations Prof: Dr. Adam Larios

Notes, books, and calculators are not authorized. Show all your work in the blank space you are given. Always justify your answer. Answers without adequate justification will not receive credit.

Some formulas that may or may not be useful:

$\mathcal{F}[f] = F(\omega) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} f(x)e^{i\omega x}\,dx$

$\mathcal{F}^{-1}[F] = f(x) = \displaystyle\int_{-\infty}^{\infty} F(\omega)e^{-i\omega x}\,d\omega$

$\nabla^2 u = \partial_{xx}u + \partial_{yy}u$ (rectangular)

$\nabla^2 u = \frac{1}{r}\partial_r(r\partial_r u) + \frac{1}{r^2}\partial_{\theta\theta}u$ (polar)

$\langle f, g\rangle \equiv (f,g) := \dfrac{1}{2\pi}\displaystyle\int_0^L f(x)g(x)\,dx$

$\|f\| \equiv \|f\|_{L^2} := \left(\displaystyle\int_0^L |f(x)|^2\,dx\right)^{1/2} \equiv \sqrt{(f,f)}$

Convolution: $(f*g)(x) = \displaystyle\int_{-\infty}^{\infty} f(y)g(x-y)\,dy$

$\mathcal{F}(f*g) = \mathcal{F}(f)\mathcal{F}(g)$

Fourier Series for $f(x)$: $A_0 + \displaystyle\sum_{n=1}^{\infty} A_n\cos\left(\tfrac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n\sin\left(\tfrac{n\pi x}{L}\right)$,

$A_0 = \dfrac{1}{2L}\displaystyle\int_{-L}^{L} f(x)\,dx, \quad A_n = \dfrac{1}{L}\int_{-L}^{L} f(x)\cos\left(\tfrac{n\pi x}{L}\right)dx, \quad B_n = \dfrac{1}{L}\int_{-L}^{L} f(x)\sin\left(\tfrac{n\pi x}{L}\right)dx$

$e^{i\theta} = \cos(\theta) + i\sin(\theta)$

Complex Fourier Series for $f(x)$: $\displaystyle\sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}$, $\quad c_n = \dfrac{1}{2L}\displaystyle\int_{-L}^{L} f(x)e^{-in\pi x/L}\,dx$

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1. **(8 points)** Suppose that $f_1(x), f_2(x), \ldots, f_5(x)$ are functions such that $\langle f_i, f_j \rangle = 0$ if $i \neq j$. Suppose another function $g(x)$ can be written as

$$g(x) = c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) + c_4 f_4(x) + c_5 f_5(x)$$

for some coefficients $c_1, \ldots, c_5$. Find c_2 in terms of g and the f_i functions (you may not need to use all the f_i functions). Your expression should not involve any of the c_i coefficients.

Apply inner-product with f_2 to both sides, and use linearity and orthogonality

$$\langle g, f_2 \rangle = c_1 \underbrace{\cancel{\langle f_1, f_2 \rangle}}_{=0} + c_2 \langle f_2, f_2 \rangle + c_3 \underbrace{\cancel{\langle f_3, f_2 \rangle}}_{=0} + c_4 \underbrace{\cancel{\langle f_4, f_2 \rangle}}_{=0} + c_5 \underbrace{\cancel{\langle f_5, f_2 \rangle}}_{=0}$$

Thus, $\langle g, f_2 \rangle = c_2 \langle f_2, f_2 \rangle$, so $c_2 = \dfrac{\langle g, f_2 \rangle}{\langle f_2, f_2 \rangle} = \dfrac{\langle g, f_2 \rangle}{\|f_2\|^2}$

2. **(12 points)**

(a) Compute the Fourier series of $f(x) = x$ on $[-L, L]$. [Hint: Using odd and even properties can simplify your work.]

$f(x) = x$ is odd, so $A_0 = 0$, $A_n = 0$.

$$B_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx = \frac{1}{L}\int_{-L}^{L} x \sin\left(\frac{n\pi x}{L}\right) dx \overset{\text{parts}}{=} -\frac{x}{\cancel{L}} \frac{\cancel{L}}{n\pi} \cos\left(\frac{n\pi x}{L}\right)\Bigg|_{-L}^{L} + \frac{1}{\cancel{L}}\int_{-L}^{L} \frac{\cancel{L}}{n\pi} \cos\left(\frac{n\pi x}{L}\right)$$

$$= \frac{-1}{n\pi}\left(L\cos(n\pi) + L\cos(n\pi)\right) = \frac{-2L}{n\pi}\cos(n\pi) = \frac{-2L}{n\pi}(-1)^n = \frac{2L}{n\pi}(-1)^{n+1}$$

Thus, the series is

$$\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) = \sum_{n=1}^{\infty} \frac{2L}{n\pi}(-1)^{n+1} \sin\left(\frac{n\pi x}{L}\right)$$

(b) Compute the complex Fourier coefficients of $g(x)$ on $[-L, L]$, where $g(x)$ is given by

$$g(x) := \begin{cases} 1, & \text{if } -L/2 \leq x \leq L/2, \\ 0, & \text{otherwise.} \end{cases}$$

$$c_n = \frac{1}{2L}\int_{-L}^{L} g(x) e^{-\frac{in\pi x}{L}} dx = \frac{1}{2L}\int_{-L/2}^{L/2} 1\, e^{-in\pi x/L} dx$$

$$= \frac{1}{2L} \frac{L}{-in\pi} e^{-in\pi x/L}\Bigg|_{-L/2}^{L/2} = \frac{-1}{n\pi} \frac{e^{\frac{-in\pi}{2}} - e^{\frac{in\pi}{2}}}{2i} = \frac{1}{n\pi} \frac{e^{\frac{in\pi}{2}} - e^{\frac{-in\pi}{2}}}{2i}$$

$$= \frac{1}{n\pi}\sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{(-1)^{n+1}}{n\pi} & \text{if } n \text{ is odd} \end{cases}$$

3. **(12 points)** Consider the heat equation $u_t = ku_{xx}$ on the infinite line $(-\infty, \infty)$, with initial condition $f(x)$. Let $\hat{u}(\omega, t)$ be the Fourier transform of $u = u(x,t)$.

   (a) Find an expression for $\hat{u}(\omega, t)$ in terms of k, ω, and $\hat{f}(\omega)$.

Note that $(u_{xx})^\wedge = iw(u_x)^\wedge = (iw)^2 \hat{u} = -w^2 \hat{u}$.

Taking the Fourier transform of both sides, and pulling out the time derivative, we have:

$$\begin{cases} \hat{u}_t = -kw^2 \hat{u} \\ \hat{u}(w,0) = \hat{f}(w) \end{cases}$$

This is an ODE in time of the form $\begin{cases} y' = cy \\ y(0) = y_0 \end{cases}$.

Thus, solution is just

$$\boxed{\hat{u}(w,t) = e^{-kw^2 t} \hat{f}(w)}$$

   (b) In class, we saw that $g(x) = \sqrt{\frac{\pi}{\beta}} e^{-x^2/4\beta}$ has a Fourier transform given by $\hat{g}(\omega) = e^{-\beta\omega^2}$. Use this fact and your expression in part (a) to express $u(x,t)$ as a convolution integral.

Let $\beta = kt$. Then $\mathcal{F}\left[\sqrt{\frac{\pi}{kt}} e^{-x^2/4kt}\right] = e^{-ktw^2} = \hat{g}(w)$. Thus, the expression in part (a) can be written $\mathcal{F}[u] = \hat{u} = \hat{g}\hat{f} = \mathcal{F}[g]\mathcal{F}[f]$

$= \mathcal{F}[f * g]$ ← convolution theorem

Taking the inverse Fourier transform, we find

$$u = u(x,t) = \mathcal{F}^{-1}(\hat{u}) = g * f = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(y) \sqrt{\frac{\pi}{kt}} e^{-(x-y)^2/4kt} \, dy.$$

4. **(10 points)** Suppose g is a differentiable function. **Using the limit definition of the derivative**, show the following identity for the convolution: $((f * g)(x))' = (f * g')(x)$ (you can assume any limits pass through the integral symbol).

$$(f * g(x))' = \frac{d}{dx}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} f(y) g(x-y)\, dy\right)$$

$$= \lim_{h \to 0} \frac{\frac{1}{2\pi}\int_{-\infty}^{\infty} f(y) g(x-y+h)\, dy - \frac{1}{2\pi}\int_{-\infty}^{\infty} f(y) g(x-y)\, dy}{h} \quad \text{by def. of derivative}$$

$$= \frac{1}{2\pi} \lim_{h\to 0} \int_{-\infty}^{\infty} f(y) \frac{g(x-y+h) - g(x-y)}{h}\, dy \quad \text{by linearity}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(y) \lim_{h\to 0} \frac{g(x-y+h) - g(x-y)}{h}\, dy \quad \text{passing limit inside (not always justified!)}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(y) g'(x-y)\, dy = (f * g')(x)$$

5. **(16 points)** Consider the following problem for the heat equation:

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + x \\ u(x,0) = f(x), \\ \frac{\partial u}{\partial x}(0,t) = 4, \\ \frac{\partial u}{\partial x}(L,t) = 7. \end{cases}$$

Note:

$\int_0^L \frac{\partial u}{\partial t}\,dx = \frac{d}{dt}\int_0^L u\,dx$ if u is smooth.

Find $\int_0^L u(x,t)\,dx$ in terms of $f(x)$ without solving for $u(x,t)$. [Note: Solving for $u(x,t)$ is not a good idea here, and it will use up your time.]

Integrate in x:

$$\frac{d}{dt}\int_0^L u\,dx = \int_0^L u_{xx}\,dx + \int_0^L x\,dx$$

Note: By fundamental thm. of Calculus:

$$\int_0^L u_{xx}\,dx = \int_0^L (u_x)_x\,dx = u_x(L) - u_x(0) = 7-4 = 3$$

$$\int_0^L x\,dx = \tfrac{1}{2}L^2$$

$$\Rightarrow \frac{d}{dt}\int_0^L u\,dx = 3 + \tfrac{1}{2}L^2$$

Integrate in time and use fund. theorem of calculus:

s is a dummy variable

$$\int_0^t\left(\frac{d}{ds}\int_0^L u(x,s)\,dx\right)ds = \left(3+\tfrac{1}{2}L^2\right)t$$

$$\Rightarrow \int_0^L u(x,t)\,dx - \int_0^L u(x,0)\,dx = \left(3+\tfrac{1}{2}L^2\right)t \Rightarrow \int_0^L u(x,t)\,dx = \int_0^L f(x)\,dx + \left(3+\frac{L^2}{2}\right)t$$

6. **(12 points)** Consider the "hyper-diffusive" heat equation with boundary conditions:

$$\begin{cases} u_t = -u_{xxxx} \\ u(0,t) = 0, \quad u(L,t) = 0, \quad u_{xx}(0,t) = 0, \quad u_{xx}(L,t) = 0. \end{cases}$$

(Initial conditions are not specified.) Show that the energy $\frac{1}{2}\|u\|^2$ is non-increasing in time.

Energy method: Take inner-product of both sides with u:

$$\langle u_t, u\rangle = -\langle u_{xxxx}, u\rangle$$

$$\Rightarrow \int_0^L u_t u\,dx = -\int_0^L u_{xxxx} u\,dx$$

$$\Rightarrow \frac{d}{dt}\left(\tfrac{1}{2}\|u\|^2\right) = -\int_0^L u_{xxxx} u\,dx$$

Note: $\frac{\partial}{\partial t}\left(\frac{1}{2}u^2\right) = u_t u$, so

$$\int_0^L u_t u\,dx = \int_0^L \frac{\partial}{\partial t}\left(\tfrac{1}{2}u^2\right)dx = \frac{d}{dt}\int_0^L \tfrac{1}{2}u^2\,dx = \frac{d}{dt}\left(\tfrac{1}{2}\|u\|^2\right)$$

Now,

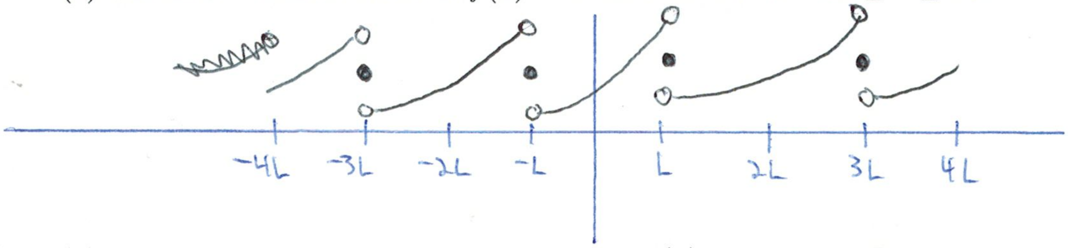
$$-\int_0^L u_{xxxx} u\,dx \overset{\text{parts}}{=} \underbrace{\cancel{-u_{xxx}u\Big|_0^L}}_{=0 \text{ since } u(0,t)=0 \text{ and } u(L,t)=0} + \int_0^L u_{xxx}u_x\,dx = \int_0^L u_{xxx}u_x\,dx$$

$$\overset{\text{parts}}{=} \underbrace{\cancel{u_{xx}}u_x\Big|_0^L}_{=0 \text{ since } u_{xx}(0,t)=0 \text{ and } u_{xx}(L,t)=0} - \int_0^L u_{xx}u_{xx}\,dx = -\int_0^L (u_{xx})^2\,dx = -\|u_{xx}\|^2.$$

Thus, $\frac{d}{dt}\left(\frac{1}{2}\|u\|^2\right) = -\|u_{xx}\|^2 \leq 0$, so energy is non-increasing defined on domain $[-L, L]$

7. **(10 points)**

(a) Sketch the Fourier series of $f(x) = e^x$ on the interval $-4L \le x \le 4L$.

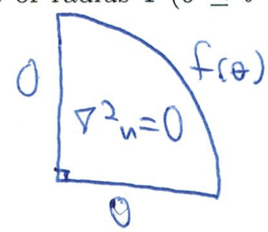


(b) Write down the value of Fourier series of $f(x) = e^x$ at $x = L$.

$$\frac{e^{L} + e^{-L}}{2}$$ (At jumps, we get the average of left & right limits)

8. **(20 points)** Solve Laplace's $\nabla^2 u = 0$ inside the quarter-circle of radius 1 ($0 \le \theta \le \pi/2$, $0 \le r \le 1$), subject to the boundary conditions

$$\frac{\partial u}{\partial \theta}(r, 0) = 0, \qquad u(r, \tfrac{\pi}{2}) = 0, \qquad u(1, \theta) = f(\theta).$$



Separate variables: $u(r,\theta) = \phi(\theta)\, G(r)$.

Then:

$$0 = \frac{1}{r}\partial_r\big(r\,\partial_r(\phi(\theta)G(r))\big) + \frac{1}{r^2}\partial_{\theta\theta}\big(\phi(\theta)G(r)\big) = \phi(\theta)\frac{1}{r}\partial_r(r\partial_r G(r)) + \frac{1}{r^2}G(r)\phi_{\theta\theta}(\theta)$$

Divide by $\frac{1}{r^2}\phi(\theta)G(r)$: $\quad \frac{1}{G} r\,\partial_r(r\partial_r G) = -\frac{\phi_{\theta\theta}}{\phi} = \lambda$ (some constant)

Two equations <u>with Boundary conditions</u>:

$$\begin{cases} r(rG')' = \lambda G \\ |G(0)| < \infty \end{cases} \qquad \begin{cases} \phi'' = -\lambda\phi \\ \phi'(0) = 0 \\ \phi(\frac{\pi}{2}) = 0 \end{cases}$$

Energy argument:
$(\phi'', \phi) = -\lambda(\phi, \phi)$
$\downarrow$ parts
$-(\phi', \phi') = -\lambda(\phi, \phi)$
$\Rightarrow \|\phi'\|^2 = \lambda\|\phi\|^2 \Rightarrow \lambda \ge 0.$

Standard solution shows $\phi(\theta) = \cos(\sqrt{\lambda}\,\theta)$
B.C.'s $\Rightarrow$ $\sqrt{\lambda} = 2n+1 \Rightarrow \phi(\theta) = \cos((2n+1)\theta)$

$\lambda \ge 0$, so two cases:

$\lambda = 0$
$\Rightarrow r(rG')' = 0$
$\Rightarrow rG' = c_1$
$\Rightarrow G' = \frac{c_1}{r}$
$\Rightarrow G' = c_2 + c_1 \ln r$
$|G(0)| < \infty \Rightarrow c_1 = 0$

$\lambda > 0$
$r(rG')' = \lambda G$
$\Rightarrow r^2 G'' + rG' - \lambda G = 0$
$\Rightarrow$ Guess $G = r^p$. Then plug in:
$r^2(p(p-1)r^{p-2}) + r(p r^{p-1}) - \lambda r^p = 0$
divide by r^p:
$p(p-1) + p - \lambda = 0 \Rightarrow p = \pm\sqrt{\lambda} = \pm(2n+1)$

$\Rightarrow G(r) = c_3 r^{2n+1} + \cancel{c_4} r^{-(2n+1)}$
$|G(0)| < \infty \Rightarrow c_4 = 0$
Product solutions: $r^{2n+1}\cos((2n+1)\theta)$

<u>Solution</u>
$$u(r,\theta) = c_2 + \sum_{n=0}^{\infty} A_{2n+1} r^{2n+1}\cos((2n+1)\theta)$$
Initial cond:
$$f(\theta) = u(1,\theta) = \sum A_{2n+1}\cos((2n+1)\theta)$$

Boundary $u(r, \frac{\pi}{2}) = 0 \Rightarrow c_2 = 0$
Thus
$$u(r,\theta) = \sum_{n=0}^{\infty} A_{2n+1} r^{2n+1}\cos((2n+1)\theta)$$
where
$$A_{2n+1} = \frac{4}{\pi}\int_0^{\pi/2} f(\theta)\cos((2n+1)\theta)\,d\theta$$