1. **(4 points)** Let f be an integrable function on $(-\infty, \infty)$. Show that for all $a, b \in \mathbb{R}$, and for all $\xi \in \mathbb{R}$, it holds that $\mathcal{F}[e^{ibx} f(ax)](\xi) = \frac{1}{a}\mathcal{F}[f]\left(\frac{\xi+b}{a}\right)$.

$$\mathcal{F}[e^{ibx} f(ax)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ibx} f(ax)\, e^{i\xi x}\, dx \qquad (1)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(ax)\, e^{i(\xi+b)x}\, dx$$

(2)

$u = ax$

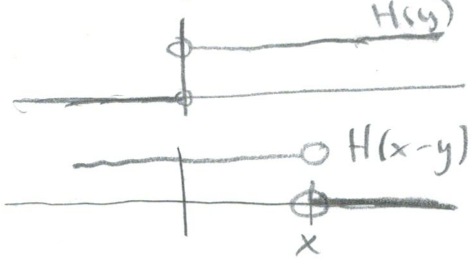
$du = a\,dx$

$\frac{1}{a}du = dx$

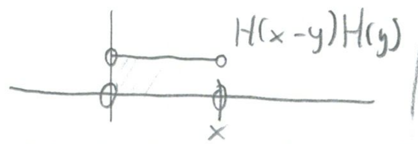
$$= \frac{1}{2\pi a}\int_{-\infty}^{\infty} f(u)\, e^{i\left(\frac{\xi+b}{a}\right)u}\, du$$

$$= \frac{1}{a}\mathcal{F}[f]\left(\frac{\xi+b}{a}\right) \qquad (1)$$

2. **(4 points)** Consider the Heaviside function $H(x)$ defined on page 1. Compute $H * H$. (Hint: draw a picture)



$$\boxed{H * H = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(x-y)\, H(y)\, dy} = \begin{cases} \frac{1}{2\pi}\int_0^x 1\,dy & x>0 \\ 0 & x<0 \end{cases}$$



$$= \begin{cases} \frac{x}{2\pi} & x>0 \\ 0 & x<0 \end{cases}$$

3. **(4 points)** Let $\delta(x)$ be the Dirac-delta function centered at $x_0 = 0$. Find $\mathcal{F}[\delta]$.

$$\mathcal{F}[\delta] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(x)\, e^{i\omega x}\, dx = \frac{1}{2\pi} e^{i\omega\cdot 0}$$

↑ by def. of Dirac δ

$$= \frac{1}{2\pi}$$

4. **(6 points)** Let $\delta(x)$ be the Dirac-delta function centered at $x_0 = 0$, and let φ be a smooth, integrable function which is zero outside of some interval. Compute

$$\int_{-\infty}^{\infty} \frac{d\delta(x)}{dx}\varphi(x)\, dx$$

Say φ is zero outside of (a, b)

(Assume the usual manipulations you know also hold for the the delta function.)

$$\int_{-\infty}^{\infty} \frac{d\delta}{dx}\varphi(x)\,dx = \int_a^b \frac{d\delta}{dx}\varphi(x)\,dx$$

parts ↓ def of Dirac-δ

$$= \underbrace{\cancel{\delta(x)\varphi(x)\Big|_a^b}}_{=0} - \int_a^b \delta(x)\varphi'(x)\,dx = -\varphi'(0)$$

3. **(4 points)** Solve the following integral equation $\int_{-\infty}^{\infty} e^{-(x-y)^2} g(y)\, dy = e^{-2x^2}$ for all $x \in (-\infty, +\infty)$, i.e., find the function g that solves the above equation.

This is Homework #6, question 52.

The left-hand side is convolution of e^{-x^2} with $g(y)$, so the equation becomes

$$e^{-x^2} * g(x) = e^{-2x^2}$$

Apply Fourier transform and use tables to find

$$\mathcal{F}[e^{-x^2} * g(x)] = \mathcal{F}[e^{-x^2}]\,\mathcal{F}[g(x)] = 2\pi \frac{1}{\sqrt{4\pi}} e^{-\frac{1}{4}w^2}\, \mathcal{F}[g(w)]$$

and $\mathcal{F}[e^{-\frac{1}{2}x^2}] = \frac{1}{\sqrt{2\pi}} e^{-w^2/2}$. Thus, $\mathcal{F}[g] = \frac{1}{\pi\sqrt{2}} e^{-w^2/4}$

Taking inverse Fourier transform gives $g(x) = \frac{\sqrt{4\pi}}{\pi\sqrt{2}} e^{-x^2} = \sqrt{\frac{2}{\pi}}\, e^{-x^2}$

4. **(5 points)** Consider the following boundary value problem:

$$u'''' = f(x), \qquad u(0) = 2, \qquad u(L) = 7. \qquad u'(0) = 1, \qquad u'(L) = 8.$$

(a) Write down the Green's function for this problem as a formula, but do not compute the coefficients. Set $G''''(x, x_0) = \delta(x - x_0)$, so that away from x_0, $G'''' = 0$, so

$$G(x, x_0) = \begin{cases} a_1 x^3 + b_1 x^2 + c_1 x + d_1, & x > x_0 \\ a_2 x^3 + b_2 x^2 + c_2 x + d_2, & x < x_0 \end{cases}$$

(b) Write down the precise equations you would use to compute the coefficients. (Hint: there should be as many equations as coefficients). **Do not compute the coefficients.**

We have 8 unknown coefficients, so we need 8 equations.

B.C.s are same on left but zero on right:

$$\left.\begin{aligned} G(0, x_0) &= 0 \\ G(L, x_0) &= 0 \\ G'(0, x_0) &= 0 \\ G'(L, x_0) &= 0 \end{aligned}\right\} \text{4 equations}$$

Since $\int_0^L \delta(x - x_0)\,dx = 1$, we get another equation:

$$G'''(L, x_0) - G'''(0, x_0) = \int_0^L G''''(x, x_0)\,dx = \int_0^L \delta = 1$$

Since $G'''' = \delta(x - x_0)$, we have $G''' = H(x - x_0)$ (Heaviside), so G'' must be continuous, and therefore so are G' and G. Thus, we get 3 more equations

$$G''(x_0^+, x_0) = G''(x_0^-, x_0)$$

$$G'(x_0^+, x_0) = G'(x_0^-, x_0)$$

$$G(x_0^+, x_0) = G(x_0^-, x_0)$$

Note: Problem 9.3.24 in the book (4th edition, maybe 5th too) is similar to this problem.

5. Suppose $\{\phi_n\}_{n=1}^{\infty}$ is an orthogonal set of smooth functions on $[0, L]$ such that any smooth, "nice enough" function g can be written as

$$g(x) = \sum_{n=1}^{\infty} a_n \phi_n(x)$$

for some constant coefficients a_n. Suppose further that ϕ_n are eigenfunctions of a linear operator L corresponding to the eigenvalues λ_n, and that all $\lambda_n < 0$ for all n.

(a) **(4 points)** How many solutions are there to $Lu = f$? Give a short explanation.

L has no zero eigenvalues, so there is exactly one solution

write $u = \sum_{n=1}^{\infty} a_n \phi_n$

(b) **(6 points)** Denote $L^2 g = L(Lg)$, and consider the problem $L^2 u = f$. Solve for the coefficients of u in terms of f, λ_n, and ϕ_n.

$$L^2 u = \sum_{n=1}^{\infty} a_n L^2 \phi_n = \sum_{n=1}^{\infty} a_n L(\lambda_n \phi_n) = \sum_{n=1}^{\infty} a_n \lambda_n^2 \phi_n$$

Thus,

$$(f, \phi_m) = (L^2 u, \phi_m) = \sum_{n=1}^{\infty} a_n \lambda_n^2 (\phi_n, \phi_m) = a_m \lambda_m^2 (\phi_m, \phi_m), \text{ so}$$

$$a_m = \frac{(f, \phi_m)}{\lambda_m^2 (\phi_m, \phi_m)} = \frac{(f, \phi_m)}{\lambda_m^2 \|\phi_m\|^2}$$

(c) **(6 points)** Consider the "heat like" equation $u_t = Lu$. (If $L = k\nabla^2$, this would be the usual heat equation.) Suppose the initial data is given by $u_0(x) = \sum_{n=1}^{\infty} b_n \phi_n(x)$. Assume the equation has a unique solution, and solve for the coefficients of u in terms of ~~k,~~ λ_n, b_n, and ϕ_n.

Write $u(x,t) = \sum_{n=1}^{\infty} a_n(t) \phi_n(x)$. Then

$$Lu = \sum_{n=1}^{\infty} a_n(t) L\phi_n = \sum_{n=1}^{\infty} a_n(t) \lambda_n \phi_n$$

$$u_t = \sum_{n=1}^{\infty} a_n'(t) \phi_n$$

Thus,

$$(Lu, \phi_m) = \sum_{n=1}^{\infty} a_n(t) \lambda_n (\phi_n, \phi_m) = a_m(t) \lambda_m \|\phi_m\|^2$$

$$\|$$

$$(u_t, \phi_m) = \sum_{n=1}^{\infty} a_n'(t) (\phi_n, \phi_m) = a_m'(t) \|\phi_m\|^2$$

so $a_m'(t) \|\phi_m\|^2 = a_m(t) \lambda_m \|\phi_m\|^2 \Rightarrow \begin{cases} a_m' = \lambda_m a_m \\ a_m(0) = b_m \leftarrow \text{from initial condition} \end{cases}$

Thus, $a_m(t) = a_m(0) e^{\lambda_m t} = b_m e^{\lambda_m t}$.

6. **(10 points)** Assume $a > b > 0$. Find a function $g(x)$ which satisfies the integral equation

$$\underbrace{\int_{-\infty}^{\infty} \frac{g(y)}{(x-y)^2+b^2}\,dy}_{= 2\pi \left(\frac{1}{x^2+b^2}\right) * g} = \frac{1}{x^2+a^2}.$$

Thus, applying $\mathcal{F}$ to both sides,

$$2\pi\, \mathcal{F}\left(\frac{1}{x^2+b^2}\right) \mathcal{F}[g] = \mathcal{F}\left[\frac{1}{x^2+a^2}\right]$$

Note that, from table

$$\mathcal{F}\left[\frac{1}{x^2+a^2}\right] = \frac{1}{2a}\mathcal{F}\left[\frac{2a}{x^2+a^2}\right] = \frac{1}{2a} e^{-a|w|}$$

Thus,

$$2\pi \frac{1}{2b} e^{-b|w|}\, \mathcal{F}[g] = \frac{1}{2a} e^{-a|w|}$$

Or, $\mathcal{F}[g] = \frac{1}{2\pi}\frac{b}{a} e^{-(a-b)|w|}$

So: $g(x) = \frac{1}{2\pi}\frac{b}{a} \cdot \frac{2(a-b)}{x^2+(a-b)^2}$

2 (2) (1) (2) (1) (2)

7. **(16 points)** Find the Green's function for the problem:

$$\begin{cases} u'' + u = f, \\ u(0) = 0, \\ \dfrac{du}{dx}(1) = 0. \end{cases}$$

Solve for G, continuous, satisfying:

$$\begin{cases} G''(x,x_0) + G(x,x_0) = \delta(x-x_0) \\ G(0,x_0) = 0 \quad (BC\ 1) \\ \dfrac{\partial G}{\partial x} G(1,x_0) = 0 \quad (BC\ 2) \end{cases}$$

Away from x_0,

$$G'' + G = 0$$

$$\Rightarrow G(x,x_0) = \begin{cases} A_1 \sin(x) + B_1\cos(x), & x < x_0 \\ A_2\sin(x) + B_2\cos(x), & x > x_0 \end{cases}$$

BC1

$$0 = G(0,x_0) = A_1\sin(0) + B_1\cos(0)$$
$$\Rightarrow B_1 = 0$$

BC2

$$0 = G'(1,x_0) = A_2\cos 1 - B_2 \sin 1$$
$$\Rightarrow A_2 = B_2 \tan 1$$

$$\Rightarrow G = \begin{cases} A_1 \sin x & x < x_0 \\ B_2(\tan 1 \sin x + \cos x), & x > x_0 \end{cases}$$

Continuity:

$$A_1 \sin x_0 = B_2(\tan 1 \sin x_0 + \cos x_0)$$
$$\Rightarrow A_1 = B_2(\tan 1 + \cot x_0)$$

$$\Rightarrow G = \begin{cases} B_2(\tan 1 + \cot x_0)\sin x, & x < x_0 \\ B_2(\tan 1 \sin x + \cos x), & x > x_0 \end{cases}$$

Integrate on $[x_0-, x_0+]$:

$$\int_{x_0^-}^{x_0^+} G''\,dx + \int_{x_0^-}^{x_0^+} G\,dx = \int_{x_0^-}^{x_0^+} \delta(x-x_0)\,dx = 1$$

$$\Rightarrow G'(x_0^+, x_0) - G'(x_0^-, x_0) + \int_{x_0^-}^{x_0^+} G\,dx = 1$$

Let $x_0^+ \to x_0$ and $x_0^- \to x_0$. Since G is continuous, $\int_{x_0^-}^{x_0^+} G\,dx \to 0$.

Thus,

$$[B_2(\cancel{\tan 1} \cos x_0 - \sin x_0)] - [B_2(\cancel{\tan 1} + \cot x_0)\cos x_0] = 1$$

$$\Rightarrow B_2 = \frac{1}{-\sin x_0 - \cot x_0 \cos x_0} \cdot \frac{\sin x_0}{\sin x_0}$$

$$= \frac{\sin x_0}{-(\sin^2 x_0 + \cos^2 x_0)} = -\sin x_0$$

$$\Rightarrow A_2 = -\sin x_0 \tan 1$$

$$A_1 = -\sin x_0 \tan 1 + \cos x_0$$

4 (2) (2) (3) (2) 4

8. Consider the equation

$$u'' + u = 1.$$

Determine the number of solutions to this equation if the boundary conditions are:

(a) **(10 points)**

$$\begin{cases} u(0) = 0, \\ u(\pi) = 0. \end{cases}$$

To apply Fredholm to operator $L = \frac{d^2}{dx^2} + 1$, we check for a zero eigenvalue. That is, we see if $Lu=0$ has a nontrivial solution:

$$Lu = 0 \Rightarrow u'' + u = 0 \Rightarrow u(x) = A\sin x + B\cos x.$$

Then

$$0 = u(0) = A\sin 0 + B\cos 0 = B \text{ , and } 0 = u(\pi) = A\sin\pi + B\cos\pi = -B.$$

Thus, $u = A\sin x$ are the only eigenfunctions with eigenvalue zero (if $A \neq 0$).

By Fredholm, we now test againts the right-hand side:

$$(A\sin x, 1) = A\int_0^\pi \sin x \cdot 1\, dx = 2A \neq 0,$$

So there are <u>no solutions</u>.

(b) **(10 points)**

$$\begin{cases} \frac{du}{dx}(0) = 0, \\ \frac{du}{dx}(\pi) = 0. \end{cases}$$

Again, $u'' + u = 0 \Rightarrow u = A\sin x + B\cos x.$

$$\frac{du}{dx} = A\cos x - B\sin x$$

$$\left.\begin{aligned} 0 &= \frac{du}{dx}(0) = A\cos 0 - B\sin 0 = A \\ 0 &= \frac{du}{dx}(\pi) = A\cos(\pi) - B\sin(\pi) = -A \end{aligned}\right\} \Rightarrow u = B\cos x$$

So, we have a zero eigenvalue. Testing...

$$(B\cos x, 1) = B\int_0^\pi \cos x \cdot 1\, dx = 0,$$

So there are infinitely many solutions.
(They happen to be of the form $u(x) = 1 + B\cos x$ for any real number B.)

9. **(20 points)** Let Ω be a three-dimensional domain, and consider the PDE

$$\nabla^2 u = f(x), \quad x \in \Omega, \quad \text{with} \quad u(x) = h(x) \quad \text{on the boundary of} \quad \Omega, \quad \text{denoted} \quad \partial\Omega.$$

Let $G(x, x_0)$ be the Green's function of this problem (the exact expression of G does not matter, just assume that G is known). Give a representation of $u(x_0)$ in terms of G, f, and h.

By definition

$$\nabla^2 G(x, x_0) = \delta(x - x_0), \quad x \in \Omega$$

and $\quad G(x, x_0) = 0 \quad , \quad x \in \partial\Omega$

Using integration by parts, (i.e., Green's formula),

$$\int_\Omega u(x) \nabla^2 G(x, x_0)\, dx = \int_\Omega (\nabla^2 u(x))\, G(x, x_0)\, dx + \int_{\partial\Omega} u(x)\, \partial_n G(x, x_0)\, d\sigma - \int_{\partial\Omega} \partial_n(u(x))\, G(x, x_0)\, d\sigma$$

Now,

$$\int_\Omega u(x) \nabla^2 G(x, x_0)\, dx = \int_\Omega u(x)\, \delta(x - x_0)\, dx = u(x_0)$$

$$\int_\Omega (\nabla^2 u(x)) G(x, x_0)\, dx = \int_\Omega f(x)\, G(x, x_0)$$

$$\int_{\partial\Omega} u(x)\, \partial_n G(x, x_0)\, d\sigma = \int_{\partial\Omega} h(x)\, \partial_n G(x, x_0)\, dx$$

and $\displaystyle \int_{\partial\Omega} \partial_n (u(x))\, G(x, x_0)\, d\sigma = 0.$

Thus,

$$u(x_0) = \int_\Omega f(x)\, G(x, x_0) + \int_{\partial\Omega} h(x)\, \partial_n (G(x, x_0))\, dx.$$