LAST NAME: Key FIRST NAME: ______________ Quiz 3

MATH 602, Differential Equations Prof: Dr. Adam Larios

Notes, books, and calculators are not authorized. Show all your work in the blank space you are given. Always justify your answer. Answers without adequate justification will not receive credit.

Notation:

$$\langle f, g\rangle \equiv (f,g) := \int_0^L f(x)g(x)\,dx, \qquad \|f\| \equiv \|f\|_{L^2} := \left(\int_0^L |f(x)|^2\,dx\right)^{1/2} \equiv \sqrt{(f,f)}.$$

1. Consider the following problem for the heat equation.

$$\begin{cases} u_t - ku_{xx} = Q(x), \\ \quad u(0,t) = 0, \qquad u(L,t) = 0, \qquad u(x,0) = u_0(x), \end{cases}$$

where $k > 0$ is a constant.

(a) **(4 points)** Show that

$$\frac{1}{2}\frac{d}{dt}\|u(\cdot,t)\|^2 + k\|u_x(\cdot,t)\|^2 = (Q,u).$$

(You don't need to justify commuting a time derivative with a space integral.)

Take inner product of heat equation with u:

$$(u_t, u) - k(u_{xx}, u) = (Q, u)$$

Now, $(u_t, u) = \int_0^L u_t u\,dx = \int_0^L \frac{1}{2}\partial_t(u^2)\,dx = \frac{1}{2}\frac{d}{dt}\int_0^L u^2\,dx = \frac{1}{2}\frac{d}{dt}\|u\|^2$

Also, from integration by parts, $=0$ since $u_x(0)u(0)=0$ and $u_x(L)u(L)=0$ from boundary conditions

$$(u_{xx}, u) = \int_0^L u_{xx} u\,dx = \cancel{u_x u\Big|_0^L} - \int_0^L u_x u_x\,dx = -\int_0^L (u_x)^2\,dx = -\|u_x\|^2$$

Thus, $\frac{1}{2}\|u\|^2 + k\|u_x\|^2 = (Q, u)$.

(b) **(3 points)** (Uniqueness) Suppose that u and v are both solutions to the above problem with the same initial data u_0. Show that $u(x,t) = v(x,t)$ for all times $t \geq 0$. (**Hint:** Consider which equation the difference $u - v$ satisfies. Use the fact that $k > 0$. There is no need for inequalities here.)

u and v are solutions to the heat equation. Thus,

$$(u-v)_t - k(u-v)_{xx} = (u_t - ku_{xx}) - (v_t - kv_{xx}) = Q - Q = 0.$$

Setting $w = u - v$, we see $w_t - kw_{xx} = 0$, and $w(0,t) = u(0,t) - v(0,t) =$

By the same argument as in 1,

$\frac{1}{2}\frac{d}{dt}\|w\|^2 + k\|w_x\|^2 = (0,u) = 0$, so $\frac{1}{2}\frac{d}{dt}\|w\|^2 \leq 0$. Integrating in time gives

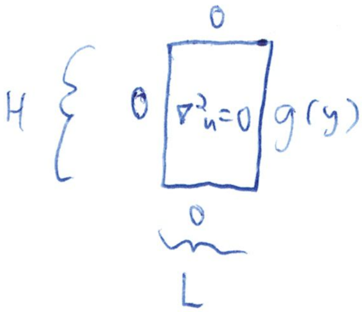
$\frac{1}{2}\|w(t)\|^2 - \frac{1}{2}\|w(0)\|^2 = \int_0^t \frac{1}{2}\frac{d}{ds}\|w(s)\|^2 ds \leq 0$, so $\|w(t)\|^2 \leq \|w(0)\|^2 = 0$, so $w = u - v =$

Orthogonality:

$$\int_0^L g(y)\sin\left(\frac{n\pi x}{L}\right)dx = B_n\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sinh\left(\frac{n\pi L}{H}\right)\sin\left(\frac{n\pi x}{L}\right)dx = B_n\cdot\frac{L}{2}\sinh\left(\frac{n\pi L}{H}\right).$$ Thus, $B_n = \frac{2}{L\sinh\left(\frac{n\pi L}{H}\right)}\int_0^L g(x)\sin\ldots$

2. **(8 points)** Solve Laplace's equation, $u_{xx} + u_{yy} = 0$, inside the rectangle $0 \le x \le L$, $0 \le y \le H$, with the following boundary conditions

$$u_x(0,y) = 0, \quad u(L,y) = g(y), \quad u(x,0) = 0, \quad u(x,H) = 0.$$



Separate variables:

$$u(x,y) = h(x)\phi(y)$$

Plug in:

$$h_{xx}\phi + h\phi_{yy} = 0 \Rightarrow \frac{1}{h}h_{xx} = -\frac{1}{\phi}\phi_{yy} = \lambda$$

function only of x — function only of y — Therefore, both are equal to a constant, say λ.

Equations with B.C.'s:

$$① \begin{cases} h_{xx} = \lambda h \\ h(0) = 0 \end{cases} \qquad ② \begin{cases} \phi_{yy} = -\lambda\phi \\ \phi(0) = 0 \\ \phi(H) = 0 \end{cases}$$

No B.C. for $h(L)$, since we must match data $g(y)$ there!!

Can λ be negative? Many way to check. Let's use energy method on ②:

$$(\phi_{yy},\phi) = \underbrace{\cancel{\phi_y\phi\Big|_0^H}}_{=0 \text{ since } \phi(0)=0\ \&\ \phi(H)=0} - \int_0^H \phi_y\phi_y\,dy = -\|\phi_y\|^2. \text{ Thus,}$$

$$-\|\phi_y\|^2 = -\lambda\|\phi\|^2, \text{ or } \quad \lambda = \frac{\|\phi_y\|^2}{\|\phi\|^2} > 0.$$

Thus, $\lambda > 0$, so solutions are

$\phi(y) = c_1\cos\sqrt{\lambda}y + c_2\sin\sqrt{\lambda}y$. Then $\phi(0)=0 \Rightarrow 0 = \phi(0) = c_1$.

And, $\phi(H) = 0 \Rightarrow 0 = c_2\sin(\sqrt{\lambda}H) \Rightarrow \sqrt{\lambda}H = n\pi$

so $\lambda = \left(\frac{n\pi}{H}\right)^2$.

Solve for h: $(\lambda > 0)$

Solutions are $h(x) = c_3\cosh(\sqrt{\lambda}x) + c_4\sinh(\sqrt{\lambda}x)$. $0 = h(0) \Rightarrow c_3 = 0$

Product solutions: $\sin\left(\frac{n\pi}{H}y\right)\sinh\left(\frac{n\pi}{H}x\right)$. Thus, $\boxed{u(x,y) = \sum_{n=1}^{\infty} B_n\sin\left(\frac{n\pi y}{H}\right)\sinh\left(\frac{n\pi x}{H}\right)}$

Now use $g(y)$:

$$g(y) = u(L,y) = \sum B_n\sin\left(\frac{n\pi x}{L}\right)\sinh\left(\frac{n\pi L}{H}\right)$$