LAST NAME: Key FIRST NAME: ______________ Quiz 4

MATH 602, Differential Equations Prof: Dr. Adam Larios

Notes, books, and calculators are not authorized. Show all your work in the blank space you are given. Always justify your answer. Answers without adequate justification will not receive credit.

Notation: $\nabla^2 u = \partial_{xx}u + \partial_{yy}u$ (rectangular), $\nabla^2 u = \frac{1}{r}\partial_r(r\partial_r u) + \frac{1}{r^2}\partial_{\theta\theta}u$ (polar)

1. Consider the following problem for Laplace's equation inside disk of radius a centered at the origin (here, $u = u(r,\theta)$ in polar coordinates).

$$\begin{cases} \nabla^2 u = 0, \\ u(a,\theta) = f(\theta), \end{cases}$$

(a) **(1 point)** Is this enough information to solve the problem uniquely? If not, what additional condition(s) should we impose?

Our choice of coordinates can cause a singularity at $r=0$. Thus, require $|u(0,\theta)| < \infty$. Also need $u(r,-\pi) = u(r,\pi)$, $\frac{\partial u}{\partial\theta}(r,-\pi) = \frac{\partial u}{\partial\theta}(r,\pi)$ (periodic)

(b) **(4 points)** Using separation of variables of the form $u(r,\theta) = \phi(\theta)G(r)$, find ordinary differential equations for ϕ and G. **Important: Make sure to include all relevant boundary conditions for ϕ and G.** There is no need to solve for ϕ and G.

Plug in: $\frac{1}{r}\frac{d}{dr}\left(r\frac{dG}{dr}\right)\phi(\theta) + \frac{1}{r^2}G(r)\frac{d^2\phi}{d\theta^2} = 0$

Divide by $\frac{1}{r^2}G(r)\phi(\theta)$:

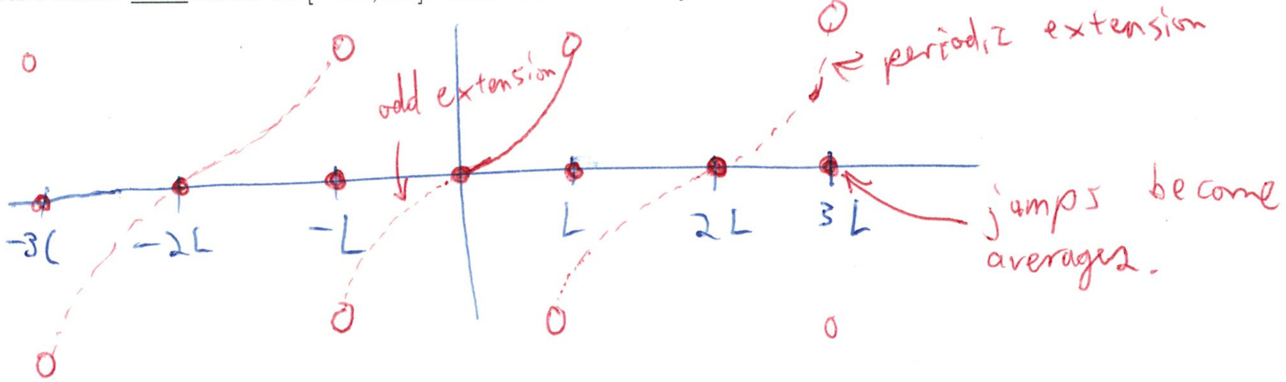
$$\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = -\frac{1}{\phi}\frac{d^2\phi}{d\theta^2} = \lambda$$

function only of r — function only of θ — therefore, both are equal to a constant, say λ.

Equations:

$$\begin{cases} \phi'' = -\lambda\phi \\ \phi(-\pi) = \phi(\pi) \\ \phi'(-\pi) = \phi(\pi) \end{cases} \qquad \begin{cases} r^2\frac{d^2G}{dr^2} + r\frac{dG}{dr} - \lambda G = 0 \\ |G(0)| < \infty \end{cases}$$

2. **(4 points)** Consider the function defined by $f(x) = x^2$, defined on $[0, L]$ Sketch the graph of its Fourier **sine** series on $[-2L, 2L]$. Make sure to clearly draw the values at $x = -2L, -L, 0, L, 2L$.



Q4

3. **(2 points)** Does the function $f(x) = x$ defined on $[-L, L]$ equal its Fourier sine series? Why or why not?

No, since $f(-L) = -L \neq L = f(L)$.

4. **(4 points)** Suppose some PDE has the general solution given by

$$u(x, y) = \sum_{n=1}^{\infty} B_n \sinh(y) \sin\left(\frac{n\pi x}{L}\right),$$

and that it comes with the boundary condition

$$u(x, L) = f(x).$$

Find B_7 in terms of $f(x)$. (You may use the facts we proved in class regarding integrals of products of sine functions without justifying them.) You answer should **not** involve n.

This is similar to quiz #3 question 2.

$$f(x) = \sum_{n=1}^{\infty} B_n \sinh(L) \sin\left(\frac{n\pi x}{L}\right)$$

Multiply by $\sin\left(\frac{7\pi x}{L}\right)$ and integrate on $[0, L]$

Since $\int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{7\pi x}{L}\right) dx = \begin{cases} 0 & \text{if } n \neq 7 \\ L/2 & \text{if } n = 7 \end{cases}$, all other terms die, and we are left with:

$$\int_0^L f(x) \sin\left(\frac{7\pi x}{L}\right) dx = B_7 \sinh(L) \cdot \frac{L}{2}$$

Thus,

$$B_7 = \frac{2}{L \cdot \sinh(L)} \cdot \int_0^L f(x) \sin\left(\frac{7\pi x}{L}\right) dx$$