LAST NAME: Key FIRST NAME: ______________ Quiz 5

MATH 602, Differential Equations Prof: Dr. Adam Larios

Notes, books, and calculators are not authorized. Show all your work in the blank space you are given. Always justify your answer. Answers without adequate justification will not receive credit.

$$\text{Fourier Transform of } f(x): \quad \mathcal{F}[f] = F(\omega) = \int_{-\infty}^{\infty} f(x)e^{i\omega x}\,dx$$

$$\text{Inverse Fourier Transform of } F(\omega): \quad \mathcal{F}^{-1}[F] = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{-i\omega x}\,d\omega$$

1. (a) **(2 points)** Consider the function $f(x) = 5 + \cos(3x) + \sin(7x)$. Find its Fourier **sine series** on the interval $[-\pi, \pi]$. (Don't confuse this with the Fourier *transform*!) **HINT:** This can be done with very little work!

Since $B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx$, and $\int_{-\pi}^{\pi}\cos(nx)\sin(mx)\,dx = 0$ for $n = 0, 1, 2, 3, \ldots$,

we have $B_0 = 0$, $B_3 = 0$, $B_7 = 1$, so the sine series is just one term:

$$\boxed{\sin(7x)}$$

(b) **(2 points)** Is the function f(x) in part (a) equal to ~~it's~~ its Fourier sine series? Justify your answer in one sentence.

Since $f(0) = 6$ and $f(\pi) = 4 \neq f(0)$, f(x) does not equal its sine series, by a theorem in section 3.3.

2. **(3 points)** Recall that if $z = a + bi$ is a complex number, then its complex conjugate is $\overline{z} = a - bi$. Let $F(\omega)$ be the Fourier transform of $f(x)$. Show that $\overline{F(\omega)} = F(-\omega)$. [Assume f(x) is real.]

$$\overline{F(\omega)} = \overline{\int_{-\infty}^{\infty} f(x)e^{i\omega x}\,dx} = \int_{-\infty}^{\infty}\overline{f(x)}\;\overline{e^{i\omega x}}\,dx.$$

$f(x)$ real $\Rightarrow \overline{f(x)} = f(x)$. Also $\overline{e^{i\omega x}} = \overline{\cos(\omega x) + i\sin(\omega x)}$

$$= \cos(\omega x) - i\sin(\omega x) = e^{-i\omega x}$$

Thus $\overline{F(\omega)} = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx = F(-\omega)$

3. **(4 points)** If $F(\omega)$ is the Fourier transform of $f(x)$, show that $-i\frac{dF}{d\omega}$ is the Fourier transform of $xf(x)$.

$$\frac{dF}{d\omega} = \frac{d}{d\omega}\int_{-\infty}^{\infty} f(x)e^{i\omega x}dx = \int_{-\infty}^{\infty} f(x)\frac{d}{d\omega}\left(e^{i\omega x}\right)dx = \int_{-\infty}^{\infty} ixf(x)e^{i\omega x}dx$$

Multiply by $-i$ (recall also $\frac{1}{i} = -i$), so that

$$-i\frac{dF}{d\omega} = \int_{-\infty}^{\infty} x f(x)e^{i\omega x}dx = \mathcal{F}(xf(x))$$

4. Consider the following heat equation problem, with periodic boundary conditions on $[-\pi, \pi]$:

$$\begin{cases} u_t = ku_{xx} \\ u(x,0) = u_0, \end{cases}$$

Suppose $u = \sum_{n=-\infty}^{\infty} c_n(t) e^{inx}$.

(a) **(2 points)** Find an **ordinary differential equation** satisfied by $c_{10}(t)$.

Plug in series to equation:

$$\sum_{n=-\infty}^{\infty} c_n'(t) e^{inx} = k \sum_{n=-\infty}^{\infty} (in)^2 c_n(t) e^{inx} = \sum_{n=-\infty}^{\infty} -n^2 k c_n(t) e^{inx}$$

Equate coefficients: $\boxed{c_n'(t) = -kn^2 c_n(t)}$ or, for $n=10$ $\boxed{c_{10}'(t) = -100k c_{10}(t)}$

(b) **(2 points)** Solve for $c_{10}(t)$ in terms of $c_{10}(0)$.

Solve the ODE:

$$c_{10}(t) = C e^{-100kt}$$

C constant to be determined

At $t=0$:

$$c_{10}(0) = C e^0, \text{ so } C = c_{10}(0)$$

So

$$\boxed{c_{10}(t) = c_{10}(0) e^{-100kt}}$$