LAST NAME: Key FIRST NAME: ____________ Quiz 6

MATH 602, Differential Equations Prof: Dr. Adam Larios

Notes, books, and calculators are not authorized. Show all your work in the blank space you are given. Always justify your answer. Answers without adequate justification will not receive credit.

$$\mathcal{F}[f] = F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{i\omega x}\,dx, \qquad \mathcal{F}^{-1}[F] = f(x) = \int_{-\infty}^{\infty} F(\omega)e^{-i\omega x}\,d\omega$$

$$(f*g)(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(y)g(x-y)\,dy \qquad \mathcal{F}(f*g) = \mathcal{F}(f)\mathcal{F}(g)$$

$$\mathcal{F}[e^{-\beta x^2}] = \frac{1}{4\pi\beta}e^{-\omega^2/4\beta}, \qquad \mathcal{F}^{-1}[e^{-\alpha\omega^2}] = \sqrt{\frac{\pi}{\alpha}}e^{-x^2/4\alpha}$$

1. **(2 points)** Evaluate

$$\int_{-\infty}^{\infty} \delta(x-2)\sqrt{x^2+1}\,dx.$$

The δ-"function" is <u>defined</u> to be something that evalutes a function when integrated against.

$$= \sqrt{2^2+1} = \sqrt{5}$$

2. **(4 points)** Suppose L is a differential operator (for example, $Lu = \frac{d^2u}{dx^2}$ or $\frac{d}{dx}\left(x\frac{du}{dx}\right) + 3u$, etc.). Consider the problem $Lu = f$ with some given boundary conditions, and suppose $G = G(x, x_0)$ is a Green's function for the problem.

   (a) Compute $LG(x, x_0)$.

   (b) Solve for u in terms of $f(x)$ and $G(x, x_0)$.

This problem is retracted. I should have said that L is symmetriy and $u(0)=0$, $u(L)=0$.

Then, we would have:

$$LG = \delta(x-x_0)$$

and

$$u = \int_0^L f(x)\,G(x,x_0)\,dx.$$

3. **(4 points)** Solve the following integral equation $\int_{-\infty}^{\infty} e^{-(x-y)^2} g(y)\, dy = e^{-2x^2}$ for all $x \in (-\infty, +\infty)$, i.e., find the function g that solves the above equation.

This is Homework #6, question 52.

The left-hand side is convolution of e^{-x^2} with $g(y)$, so the equation becomes

$$e^{-x^2} * g(x) = e^{-2x^2}$$

Apply Fourier transform and use tables to find

$$\mathcal{F}[e^{-x^2} * g(x)] = \mathcal{F}[e^{-x^2}]\,\mathcal{F}[g(x)] = 2\pi \frac{1}{\sqrt{4\pi}} e^{-\frac{1}{4}w^2}\,\mathcal{F}[g(w)]$$

and $\mathcal{F}[e^{-\frac{1}{2}x^2}] = \frac{1}{\sqrt{2\pi}} e^{-w^2/2}$. Thus, $\mathcal{F}[g] = \frac{1}{\pi\sqrt{2}} e^{-w^2/4}$

Taking inverse Fourier transform gives $g(x) = \frac{\sqrt{4\pi}}{\pi\sqrt{2}} e^{-x^2} = \sqrt{\frac{2}{\pi}}\, e^{-x^2}$

4. **(5 points)** Consider the following boundary value problem:

$$u'''' = f(x), \qquad u(0) = 2, \qquad u(L) = 7. \qquad u'(0) = 1, \qquad u'(L) = 8.$$

(a) Write down the Green's function for this problem as a formula, but do not compute the coefficients. Set $G''''(x, x_0) = \delta(x - x_0)$, so that away from x_0, $G'''' = 0$, so

$$G(x, x_0) = \begin{cases} a_1x^3 + b_1x^2 + c_1x + d_1, & x > x_0 \\ a_2x^3 + b_2x^2 + c_2x + d_2, & x < x_0 \end{cases}$$

(b) Write down the precise equations you would use to compute the coefficients. (Hint: there should be as many equations as coefficients). **Do not compute the coefficients.**

We have 8 unknown coefficients, so we need 8 equations.

B.C.s are same on left but zero on right:

$$\left.\begin{aligned} G(0, x_0) &= 0 \\ G(L, x_0) &= 0 \\ G'(0, x_0) &= 0 \\ G'(L, x_0) &= 0 \end{aligned}\right\} \text{4 equations}$$

Since $\int_0^L \delta(x-x_0)\,dx = 1$, we get another equation:

$$G'''(L, x_0) - G'''(0, x_0) = \int_0^L G''''(x, x_0)\,dx = \int_0^L \delta = 1$$

Since $G'''' = \delta(x - x_0)$, we have $G''' = H(x - x_0)$ (Heaviside), so G'' must be continuous, and therefore so are G' and G. Thus, we get 3 more equations

$$G''(x_0^+, x_0) = G''(x_0^-, x_0)$$

$$G'(x_0^+, x_0) = G'(x_0^-, x_0)$$

$$G(x_0^+, x_0) = G(x_0^-, x_0)$$

Note: Problem 9.3.24 in the book (4th edition, maybe 5th too) is similar to this problem.