LAST NAME: Key FIRST NAME: Quiz 8

MATH 602, Differential Equations Prof: Dr. Adam Larios

Notes, books, and calculators are not authorized. Show all your work in the blank space you are given. Always justify your answer. Answers without adequate justification will not receive credit.

1. **(3 points)** Suppose that $\partial_t\rho + \partial_x(q(\rho)) = 0$ on the domain $[0, L]$, $t > 0$, under the periodic boundary conditions $\rho(0,t) = \rho(L,t)$. Assuming that all the functions involved are "nice" (e.g., smooth and bounded), show that $\rho(x,t)$ is a conserved quantity, that is, $\frac{d}{dt}\int_0^L \rho(x,t)\,dx = 0$.

this was done in the class notes in the n-dimensional case, $\partial_t\rho + \nabla\cdot q = 0$. Just integrate in space and use fund. theorem of calculus (F.T.C.):

$$0 = \int_0^L (\partial_t\rho + \partial_x(q(\rho)))dx = \frac{d}{dt}\int_0^L \rho\,dx + \int_0^L \frac{\partial}{\partial x} q(\rho(x,t))dx$$

$$\overset{\text{FTC}}{=} \frac{d}{dt}\int_0^L \rho\,dx + \cancel{q(\rho(L,t)) - q(\rho(0,t))} = \frac{d}{dt}\int_0^L \rho\,dx.$$

cancel since $\rho(0,t) = \rho(L,t)$

2. **(6 points)** Consider the following conservation equation:

$$\partial_t\rho + \partial_x(q(\rho)) = 0 \quad x \in (-\infty,\infty), \ t > 0 \qquad \rho(x,0) = \rho_0(x) := \begin{cases} \frac{1}{8} & \text{if } x < 0, \\ \frac{1}{4} & \text{if } x > 0 \end{cases}$$

where $q(\rho) = \rho\cdot(2 - 3\rho)$ (and $\rho(x,t)$ is the conserved quantity).

(a) Given that the data produces a shock, find the speed of this shock.

$$\frac{dx_s}{dt} = \frac{q(\rho^-) - q(\rho^+)}{\rho^- - \rho^+} = \frac{q(\frac{1}{8}) - q(\frac{1}{4})}{\frac{1}{8} - \frac{1}{4}} = \frac{\frac{1}{8}(2 - \frac{3}{8}) - \frac{2}{8}(2 - \frac{6}{8})}{\frac{1}{8} - \frac{2}{8}} = \frac{\frac{13}{8} - 2(\frac{10}{8})}{-1} = +\frac{7}{8}$$

Also, note that $x_s(0) = 0$, so $x_s(t) = \frac{7}{8}t$

(b) Find the solution to the problem.

We set $\frac{dx}{dt} = q'(\rho) = 2 - 6\rho$. By chain rule, ρ is constant on these curves. Thus, $\frac{dx}{dt} = 2 - 6\rho(x_0, 0)$, so

$$x = x(t) = (2 - 6\rho(x_0,0))t + x_0 = \begin{cases} (2 - \frac{6}{8})t + x_0, & x_0 < 0 \\ (2 - \frac{6}{4})t + x_0, & x_0 > 0 \end{cases} = \begin{cases} \frac{5}{4}t + x_0, & x_0 < 0 \\ \frac{1}{2}t + x_0, & x_0 > 0 \end{cases}$$

or, $x_0 = x - \frac{5}{4}t$ if $x < x_s$

and $x_0 = x - \frac{1}{2}t$ if $x > x_s$

Thus,

$$\rho(x,t) = \begin{cases} \rho_0(x - \frac{5}{4}t), & x < x_s \\ \rho_0(x - \frac{1}{2}t), & x > x_s \end{cases} = \begin{cases} \frac{1}{8}, & x < \frac{7}{8}t \\ \frac{1}{4}, & x > \frac{7}{8}t \end{cases}$$

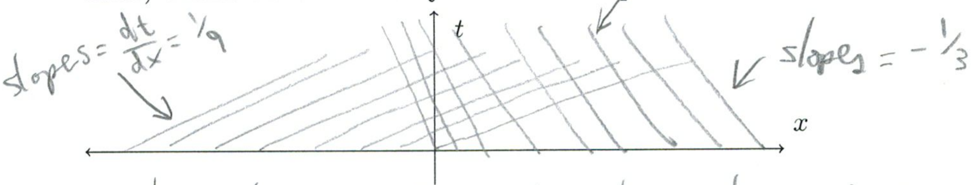
3. Consider the conservation equation:

$$\partial_t \rho + \partial_x(q(\rho)) = 0, \qquad x \in (-\infty, \infty),\ t > 0 \qquad \rho(x,0) = \rho_0(x) := \begin{cases} 1 & \text{if } x < 0, \\ -1 & \text{if } x > 0 \end{cases}$$

where $q(\rho) = \rho^6 + 3\rho$.

picture not correct here. Should be redrawn once we calculate the shock.

(a) **(3 points)** Draw a picture of the characteristic lines. (This requires calculation of the lines.) **Make sure to clearly label the slopes.**



Set $\frac{dx}{dt} = q'(\rho) = 6\rho^5 + 3$. By chain rule, ρ is constant on these curves, so $\frac{dx}{dt} = 6(\rho(x_0,0))^5 + 3$, or $x(t) = (6(\rho(x_0,0))^5 + 3)t + x_0$.

If $x_0 < 0$, $x(t) = (6+3)t + x_0 = 9t + x_0$. If $x_0 > 0$, $x(t) = (6(-1)^5 + 3)t + x_0 = -3t + x_0$

Thus, $\frac{dx}{dt} = \begin{cases} 9, & x < 0 \\ -3, & x > 0 \end{cases}$ (not accounting for the shock), or $\frac{dt}{dx} = \begin{cases} 1/9, & x < 0 \\ -1/3, & x > 0 \end{cases}$

4. **(3 points)** Is there a shock or an expansion wave here? Give a one-sentence justification of your answer by referencing the characteristics. Find the equation of the location of the shock, or the equations of the fan-wave expansion, depending on which one it is.

Characteristics intersect (see picture) so there is a shock.

By Rankine-Hugoniot, shock velocity $x_s(t)$ satisfies:

$$\frac{dx_s}{dt} = \frac{q(\rho^-) - q(\rho^+)}{\rho^- - \rho^+} = \frac{[(+1)^6 + 3(+1)] - [(-1)^6 + 3(-1)]}{(+1) - (-1)}$$

$$= \frac{(4) - (-2)}{2} = 3.$$

(Note that this means $\frac{dt}{dx_s} = \frac{1}{3}$)

Thus, the picture is actually: