LAST NAME: Key FIRST NAME: Quiz 9

MATH 602, Differential Equations Prof: Dr. Adam Larios

Notes, books, and calculators are not authorized. Show all your work in the blank space you are given. Always justify your answer. Answers without adequate justification will not receive credit.

1. **(3 points)** Consider the wave equation on the line $(-\infty, \infty)$, in the form:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad u(x,0) = f(x), \qquad \frac{\partial u}{\partial t}(x,0) = g(x).$$

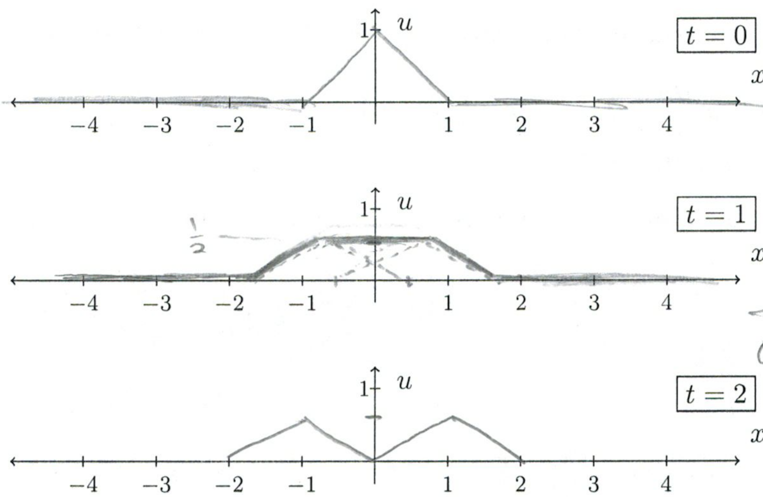
Recall d'Alembert's 1747 solution:

$$u(x,t) = F(x-ct) + G(x+ct), \quad F(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(y)\,dy, \quad G(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(y)\,dy.$$

Let $c = 0.5$. Sketch the graphs of $u(x,t)$ for $t=0$, $t=1$, $t=2$, given initial data:

$$u(x,0) = \begin{cases} 1-|x| & \text{if } |x|<1, \\ 0 & \text{otherwise,} \end{cases} \qquad \frac{\partial u}{\partial t}(x,0) = 0.$$

$g = 0$, so $u(x,t) = \frac{1}{2}f(x-0.5t) + \frac{1}{2}f(x+0.5t)$



splits into the two dotted curves, which are added together

curves continue to move apart

2. **(6 points)** Consider the conservation equation:

$$\partial_t \rho + \rho^4 \partial_x \rho, \qquad x \in (-\infty,\infty),\ t>0 \qquad \rho(x,0) = \rho_0(x) := \begin{cases} -11 & \text{if } x<0, \\ 2 & \text{if } x>0 \end{cases}$$

(so that $q = q(\rho) = \frac{1}{5}\rho^5$).

(a) Find the (family of) equations of the characteristics for the fan expansion wave (i.e., corresponding to the density values ρ_{fan}).

$\frac{dx}{dt} = \rho^4$. At expansion wave: $\frac{dx}{dt} = \rho_0^4$, $-11 < \rho_0 < 2$,

so $x(t) = \rho_0^4 t + x_0$. Since expansion wave is at $x_0 = 0$, curves are

$$\boxed{x = \rho_0^4 t, \ -11 < \rho_0 < 2}$$

(b) (Continued from previous page.) Find a **differential equation** satisfied by the shock velocity $x_s(t)$. **Do not try to solve this equation!**

$$\frac{dx_s}{dt} = \frac{q(\rho^+) - q(\rho^-)}{\rho^+ - \rho^-} \quad \text{where} \quad q = \frac{1}{5}\rho^5.$$

To left of shock, $\rho^- = -11$. To right of shock, we are in the expansion wave, where $\rho^+ = \left(\frac{x}{t}\right)^{1/4}$.

Thus,

$$\frac{dx_s}{dt} = \frac{\frac{1}{5}\left(\left(\frac{x_s}{t}\right)^{1/4}\right)^5 - \frac{1}{5}(-11)^5}{\left(\frac{x_s}{t}\right)^{1/4} - (-11)} = \frac{1}{5}\,\frac{11^5 + \left(\frac{x_s}{t}\right)^{5/4}}{11 + \left(\frac{x_s}{t}\right)^{1/4}}$$

3. **(6 points)** Determine a parametric representation of the solution of

$$\begin{cases} \dfrac{\partial \rho}{\partial t} - \rho^2 \dfrac{\partial \rho}{\partial x} = 3\rho, & x \in (-\infty, \infty) \\ \rho(x,0) = \rho_0(x) \end{cases}$$

in terms of ρ_0. **Hint:** Use methods we learned in class to reduce the problem to ODEs.

Set $\frac{dx}{dt} = -\rho^2$. Then, $\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \frac{dx}{dt}\frac{\partial \rho}{\partial x} = 3\rho$

chain rule

from equation.

We don't know ρ, so we can't solve $\frac{dx}{dt} = \rho$ yet. Thus, solve $\frac{d\rho}{dt} = 3\rho$ to get $\rho(x(t),t) = e^{3t}\rho(x_0,0) = \rho_0(x_0)e^{3t}$.

constant, from initial condition

with $x(0) = x_0$.

Now, put this back into other equation:

$$\frac{dx}{dt} = -\rho^2 = -\left(\rho_0(x_0)e^{3t}\right)^2 = -\rho_0^2(x_0)\, e^{6t}, \quad \text{with } x(0) = x_0.$$

Solve:

$x = x(t) = x_0 e^{6t}$, or $x_0 = x e^{-6t}$. Thus,

$$\boxed{\rho(x,t) = \rho_0(xe^{-6t})\, e^{3t}}$$