M602: Methods and Applications of Partial Differential Equations. Final TEST, Dec, 2013. Notes, books, and calculators are not authorized. Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with no justification will not be graded.

Here are some formulae that you may want to use:

$$\mathcal{F}(f)(\omega) \stackrel{\text{def}}{=} \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x)e^{i\omega x}\mathrm{d}x, \qquad \mathcal{F}^{-1}(f)(x) = \int_{-\infty}^{+\infty} f(\omega)e^{-i\omega x}\mathrm{d}\omega, \qquad \mathcal{F}(f*g) = 2\pi\mathcal{F}(f)\mathcal{F}(g), \tag{1}$$

$$\mathcal{F}(S_\lambda(x)) = \frac{1}{\pi}\frac{\sin(\lambda\omega)}{\omega}, \quad \text{where} \quad S_\lambda(x) = \begin{cases} 1 & \text{if } |x| \le \lambda \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

The implicit representation of the solution to the equation $\partial_t v + \partial_x q(v) = 0$, $v(x,0) = v_0(x)$, is

$$X(s,t) = q'(v_0(s))t + s; \quad v(X(s,t),t) = v_0(s). \tag{3}$$

Question 1: Consider the telegraph equation $\partial_{tt}u + (\alpha+\beta)\partial_t u + \alpha\beta u - c^2\partial_{xx}u = 0$ with $\alpha,\beta \ge 0$, $u(x,0) = f(x)$, $\partial_t u(x,0) = g(x)$, $x \in \mathbb{R}$, $t > 0$ and boundary condition at infinity $u(\pm\infty, t) = 0$. Show that $E(t) := \int_{-\infty}^{+\infty} \frac{1}{2}\left((\partial_t u(x,t))^2 + c^2(\partial_x u(x,t))^2 + \alpha\beta(u(x,t))^2\right)\mathrm{d}x$ decreases in time. (Hint: Energy argument with $\partial_t u$.)

Solution: We test the equation with $\partial_t u$ and integrate over $\mathbb{R}$,

$$\int_{\mathbb{R}} \left(\partial_t(\frac{1}{2}(\partial_t u)^2) + (\alpha+\beta)(\partial_t u)^2 + \alpha\beta\partial_t(\frac{1}{2}u^2)\right)\mathsf{d}x - c^2\int_{\mathbb{R}} \partial_{xx}u(x,t)\partial_t u(x,t)\mathsf{d}x = 0,$$

where we used the product rule, $2\psi(\gamma)\partial_\gamma\psi(\gamma) = \partial_\gamma(\psi^2(\gamma))$. Integration by parts in the last integral gives

$$\int_{\mathbb{R}} \left(\partial_t(\frac{1}{2}(\partial_t u)^2) + (\alpha+\beta)(\partial_t u)^2 + \alpha\beta\partial_t(\frac{1}{2}u^2)\right)\mathsf{d}x + c^2\int_{\mathbb{R}} \partial_x u(x,t)\partial_t\partial_x u(x,t)\mathsf{d}x = 0,$$

where we used the boundary condition at infinity $u(\pm\infty,t) = 0$. This means that

$$\int_{\mathbb{R}} \left(\partial_t(\frac{1}{2}(\partial_t u)^2) + (\alpha+\beta)(\partial_t u)^2 + \alpha\beta\partial_t(\frac{1}{2}u^2) + c^2\partial_t(\frac{1}{2}(\partial_x u)^2)\right)\mathsf{d}x = 0.$$

After exchanging the time derivative and the space integral, we have

$$\partial_t\int_{\mathbb{R}} \left(\frac{1}{2}(\partial_t u)^2 + \alpha\beta\frac{1}{2}u^2 + c^2\frac{1}{2}(\partial_x u)^2\right)\mathsf{d}x = -(\alpha+\beta)\int_{\mathbb{R}}(\partial_t u)^2\mathsf{d}x,$$

which means

$$\partial_t E(t) = -(\alpha+\beta)\int_{\mathbb{R}}(\partial_t u)^2\mathsf{d}x \le 0,$$

i.e., $E(t)$ decreases in time.

Question 2: Let d be positive integer and let $u : \mathbb{R}^d\times\mathbb{R}_+ \longrightarrow \mathbb{R}$ be the solution to the nonlinear PDE $\partial_t u(\boldsymbol{x},t) - a\Delta u(\boldsymbol{x},t) + b(\nabla u(\boldsymbol{x},t))\cdot(\nabla u(\boldsymbol{x},t)) = 0$, with $u(\boldsymbol{x},0) = f(\boldsymbol{x})$, where $a > 0$ and $b \neq 0$ are real numbers, and $f(\boldsymbol{x})$ is a smooth function. Let $v(\boldsymbol{x},t) = \mathrm{e}^{-\frac{b}{a}u(\boldsymbol{x},t)}$. (i) Compute $\nabla v(\boldsymbol{x},t)$ and $\nabla\cdot(\nabla v(\boldsymbol{x},t))$ (Hint: compute $\partial_{x_i} v(\boldsymbol{x},t)$ for $i = 1,\dots,d$, then $\sum_{i=1}^d \partial_{x_i x_i} v(\boldsymbol{x},t)$. It is the same question as question 1 in mdt1; it is worded differently.)

Solution: The chain rule gives

$$\partial_{x_i} v(\boldsymbol{x},t) = -\frac{b}{a}\mathsf{e}^{-\frac{b}{a}u(\boldsymbol{x},t)}\partial_{x_i} u(\boldsymbol{x},t),$$

and

$$\sum_{i=1}^d \partial_{x_i x_i} v(\boldsymbol{x},t) = -\frac{b}{a}\mathsf{e}^{-\frac{b}{a}u(\boldsymbol{x},t)}\sum_{i=1}^d \partial_{x_i x_i} u(\boldsymbol{x},t) + \frac{b^2}{a^2}\mathsf{e}^{-\frac{b}{a}u(\boldsymbol{x},t)}\sum_{i=1}^d \partial_{x_i} u(\boldsymbol{x},t)\partial_{x_i} u(\boldsymbol{x},t).$$

This implies that

$$\nabla v(\boldsymbol{x},t) = -\frac{b}{a}\mathsf{e}^{-\frac{b}{a}u(\boldsymbol{x},t)}\nabla u(\boldsymbol{x},t),$$

and

$$\Delta v(\boldsymbol{x},t) = -\frac{b}{a}\mathsf{e}^{-\frac{b}{a}u(\boldsymbol{x},t)}\Delta u(\boldsymbol{x},t) + \frac{b^2}{a^2}\mathsf{e}^{-\frac{b}{a}u(\boldsymbol{x},t)}(\nabla u(\boldsymbol{x},t))\cdot(\nabla u(\boldsymbol{x},t)).$$

(ii) Compute $\partial_t v(\boldsymbol{x},t)$, $\partial_t v(\boldsymbol{x},t) - a\Delta v(\boldsymbol{x},t)$, and $v(\boldsymbol{x},0)$.

Solution: Using again the chain rule we obtain

$$\partial_t v(\boldsymbol{x},t) = -\frac{b}{a}\mathsf{e}^{-\frac{b}{a}u(\boldsymbol{x},t)}\partial_t u(\boldsymbol{x},t).$$

Combining the above results gives

$$\partial_t v(\boldsymbol{x},t) - a\Delta v(\boldsymbol{x},t) = -\frac{b}{a}\mathsf{e}^{-\frac{b}{a}u(\boldsymbol{x},t)}\left(\partial_t u(\boldsymbol{x},t) - a\Delta u(\boldsymbol{x},t) + b(\nabla u(\boldsymbol{x},t))\cdot(\nabla u(\boldsymbol{x},t))\right) = 0$$

i.e., v solves the heat equation $\partial_t v(\boldsymbol{x},t) - a\Delta v(\boldsymbol{x},t) = 0$ with initial condition $v(\boldsymbol{x},0) = \mathsf{e}^{-\frac{b}{a}f(\boldsymbol{x})}$.

(iii) Recall that the solution to the heat equation $\partial_t \phi(\boldsymbol{x},t) - a\Delta\phi(\boldsymbol{x},t) = 0$ with initial condition $\phi(\boldsymbol{x},0) = \psi_0(\boldsymbol{x})$ is given by $\psi_0(\boldsymbol{x})(4\pi a t)^{-d/2}\int_{\mathbb{R}^d}\mathrm{e}^{-\frac{\|\boldsymbol{x}-\boldsymbol{y}\|^2}{4at}}\mathrm{d}\boldsymbol{y}$. Give the integral representation of $u(\boldsymbol{x},t)$ in terms of a, b, and f.

Solution: Since $\partial_t v(\boldsymbol{x},t) - a\Delta v(\boldsymbol{x},t) = 0$ with initial condition $v(\boldsymbol{x},0) = \mathsf{e}^{-\frac{b}{a}f(\boldsymbol{x})}$, the integral representation of the function u is then given by

$$u(\boldsymbol{x},t) = -\frac{a}{b}\log(v(\boldsymbol{x},t)), \quad \text{with,} \quad v(\boldsymbol{x},t) = \mathsf{e}^{-\frac{b}{a}f(\boldsymbol{x})}(4\pi a t)^{-d/2}\int_{\mathbb{R}^d}\mathsf{e}^{-\frac{\|\boldsymbol{x}-\boldsymbol{y}\|^2}{4at}}\mathsf{d}\boldsymbol{y}$$

Question 3: Consider the telegraph equation $\partial_{tt}u + 2\alpha\partial_t u + \alpha^2 u - c^2\partial_{xx}u = 0$ with $\alpha \geq 0$, $u(x,0) = 0$, $\partial_t u(x,0) = g(x)$, $x \in \mathbb{R}$, $t > 0$ and boundary condition at infinity $u(\pm\infty, t) = 0$. Solve the equation by the Fourier transform technique. (Hint: the solution to the ODE $\phi''(t) + 2\alpha\phi'(t) + (\alpha^2 + \lambda^2)\phi(t) = 0$ is $\phi(t) = \mathrm{e}^{-\alpha t}(a\cos(\lambda t) + b\sin(\lambda t))$)

Solution: Applying the Fourier transform with respect to x to the equation, we infer that

$$\begin{aligned} 0 &= \partial_{tt}\mathcal{F}(u)(\omega,t) + 2\alpha\partial_t\mathcal{F}(u)(\omega,t) + \alpha^2\mathcal{F}(u)(\omega,t) - c^2(-i\omega)^2\mathcal{F}(u)(\omega,t) \\ &= \partial_{tt}\mathcal{F}(u)(\omega,t) + 2\alpha\partial_t\mathcal{F}(u)(\omega,t) + (\alpha^2 + c^2\omega^2)\mathcal{F}(u)(\omega,t) \end{aligned}$$

Using the hint, we deduce that

$$\mathcal{F}(u)(\omega,t) = \mathsf{e}^{-\alpha t}(a(\omega)\cos(\omega ct) + b(\omega)\sin(\omega ct)).$$

The initial condition implies that $a(\omega) = 0$ and $\mathcal{F}(g)(\omega) = \omega c b(\omega)$; as a result, $b(\omega) = \mathcal{F}(g)(\omega)/(\omega c)$ and

$$\mathcal{F}(u)(\omega,t) = \mathsf{e}^{-\alpha t}\mathcal{F}(g)\frac{\sin(\omega ct)}{\omega c}.$$

Then using (2), we have

$$\mathcal{F}(u)(\omega,t) = \frac{\pi}{c}\mathsf{e}^{-\alpha t}\mathcal{F}(g)\mathcal{F}(S_{ct}).$$

The convolution theorem implies that

$$u(x,t) = \mathsf{e}^{-\alpha t}\frac{1}{2c}g * S_{ct} = \mathsf{e}^{-\alpha t}\frac{1}{2c}\int_{-\infty}^{\infty} g(y)S_{ct}(x-y)\mathsf{d}y.$$

Finally the definition of S_{ct} implies that $S_{ct}(x-y)$ is equal to 1 if $-ct < x - y < ct$ and is equal zero otherwise, which finally means that

$$u(x,t) = \mathsf{e}^{-\alpha t}\frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\mathsf{d}y.$$

Question 4: Consider the equation $-\partial_x((1+x)\partial_x u(x)) + \partial_x u(x) = f(x)$, $x \in (0,1)$ with $u(0) = \alpha$ and $-2\partial_x u(1) + u(1) = \beta$. Let $G(x, x_0)$ be the Green's function. (i) Give the integral representation of $u(x_0)$ for all $x_0 \in (0,1)$ in terms of G, f, α and β and give the equation and boundary conditions that G must satisfy. Do not compute G at this question. (Hint: The differential operator is not self-adjoint. You should find $G(0, x_0) = 0$, $\partial_x G(1, x_0) = 0$).

Solution: We multiply the PDE by $G(x, x_0)$ and integrate by parts,

$$\begin{aligned}\int_0^1 f(x)G(x,x_0)\mathsf{d}x &= \int_0^1 ((1+x)\partial_x u(x)\partial_x G(x,x_0) - u(x)\partial_x G(x,x_0))\,\mathsf{d}x + [(-(1+x)\partial_x u(x) + u(x))G(x,x_0)]_0^1 \\ &= \int_0^1 ((1+x)\partial_x u(x)\partial_x G(x,x_0) - u(x)\partial_x G(x,x_0))\,\mathsf{d}x + \beta G(1,x_0) - (-\partial_x u(0) + \alpha)G(0,x_0).\end{aligned}$$

Since $\partial_x u(0)$ is not known, we must have $G(0, x_0) = 0$. Then

$$\begin{aligned}\int_0^1 f(x)G(x,x_0)\mathsf{d}x &= \int_0^1 (-u(x)\partial((1+x)\partial_x G(x,x_0)) - u(x)\partial_x G(x,x_0))\,\mathsf{d}x + \beta G(1,x_0) + [(1+x)u(x)\partial_x G(x,x_0)]_0^1 \\ &= \int_0^1 -u(x)\left(\partial((1+x)\partial_x G(x,x_0)) + \partial_x G(x,x_0)\right)\mathsf{d}x + \beta G(1,x_0) + 2u(1)\partial_x G(1,x_0) - \alpha\partial_x G(0,x_0).\end{aligned}$$

Since $u(1)$ is not known, we must have $\partial_x G(1, x_0) = 0$. Finally G must satisfy

$$-\partial_x((1+x)\partial_x G(x,x_0)) - \partial_x G(x,x_0) = \delta(x-x_0), \quad G(0,x_0) = 0, \quad \partial_x G(1,x_0) = 0,$$

and we have the following representation for $u(x_0)$,

$$u(x_0) = \int_0^1 f(x)G(x,x_0)\mathsf{d}x - \beta G(1,x_0) + \alpha\partial_x G(0,x_0).$$

(ii) Compute $G(x,x_0)$ such that $-\partial_x((1+x)\partial_x G(x,x_0)) - \partial_x G(x,x_0) = \delta(x-x_0)$, $G(0,x_0)=0$, $\partial_x G(1,x_0)=0$, for all $x, x_0 \in (0,1)$. (Hint: observe that $(1+x)\phi'(x)+\phi(x) = ((1+x)\phi(x))'$.)

Solution: For all $x \neq x_0$ we have

$$-\partial((1+x)\partial_x G(x,x_0)) - \partial_x G(x,x_0) = -\partial((1+x)\partial_x G(x,x_0) + G(x,x_0)) = 0$$

Using the hint, this implies that

$$(1+x)\partial_x G(x,x_0) + G(x,x_0) = \partial_x((1+x)G(x,x_0)) = a,$$

In conclusion

$$G(x,x_0) = \begin{cases} \frac{ax+b}{1+x} & \text{if } x \leq x_0 \\ \frac{cx+d}{1+x} & \text{if } x_0 \leq x. \end{cases}$$

The boundary condition at 0 gives

$$G(0,x_0) = 0 = b,$$

i.e., $b=0$, $G(x,x_0) = \frac{ax}{1+x}$ if $x \leq x_0$. The boundary condition at 1 gives

$$\partial_x G(0,x_0) = \frac{c(1+0) - (c\times 0 + d)\times 1}{(1+0)^2} = 0,$$

i.e., $c=d$. $G(x,x_0)=c$, if $x_0 \leq x$. We need to impose the continuity of $G(x,x_0)$ at x_0,

$$\frac{ax_0}{1+x_0} = c,$$

which gives $ax_0 = c(1+x_0)$. The jump condition gives

$$\begin{aligned} 1 &= \lim_{\epsilon\to 0}\int_{x_0-\epsilon}^{x_0+\epsilon} (-\partial_x((1+x)\partial_x G(x,x_0)) + \partial_x G(x,x_0))\,\mathsf{d}x = (1+x_0)(\partial_x G(x_0^-,x_0) - \partial_x G(x_0^+,x_0)) \\ &= (1+x_0)(\frac{a(1+x_0)-ax_0}{(1+x_0)^2}) = \frac{a}{1+x_0}. \end{aligned}$$

In conclusion $a=(1+x_0)$ and $c=x_0$, Then

$$G(x,x_0) = \begin{cases} \frac{x(1+x_0)}{1+x} & \text{if } x \leq x_0 \\ x_0 & \text{if } x_0 \leq x. \end{cases}$$

Question 5: Consider the conservation equation $\partial_t \rho + \partial_x(\sin(\frac{\pi}{2}\rho)) = 0$, $x \in \mathbb{R}$, $t > 0$, with initial data $\rho_0(x) = 0$ if $x < 0$ and $\rho_0(x) = 1$ if $x > 0$. Draw the characteristics and give the explicit representation of the solution.

Solution: The implicit representation of the solution to the equation $\partial_t \rho + \partial_x q(\rho) = 0$, $\rho(x, 0) = \rho_0(x)$, is

$$X(s,t) = q'(\rho_0(s))t + s; \quad \rho(X(s,t),t) = \rho_0(s). \tag{4}$$

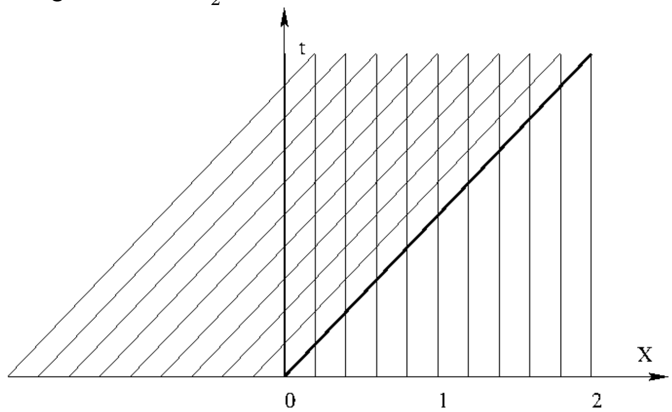
The explicit representation is obtained by expressing s in terms of X and t.
Case 1: $s < 0$, we have $\rho_0(s) = 0$, $q'(\rho_0(s)) = \frac{\pi}{2}\cos(0) = \frac{\pi}{2}$, $X = \frac{\pi}{2}t + s$, which means $s = X - \frac{\pi}{2}t$. Then

$$\rho(x,t) = 0 \text{ if } x < \frac{\pi}{2}t.$$

Case 2: $0 < s$, we have $\rho_0(s) = 1$, $q'(\rho_0(s)) = \frac{\pi}{2}\cos(\frac{\pi}{2}) = 0$, $X = s$. Then

$$\rho(x,t) = 1 \text{ if } 0 < x.$$

The characteristics cross in the region $0 < x < \frac{\pi}{2}t$; this means that there is a shock in this region.

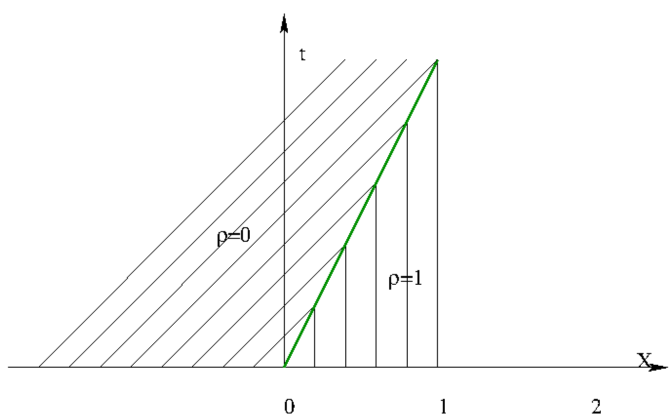


The Rankin-Hugoniot formula gives the speed of the shock

$$\frac{\mathrm{d}x_s(t)}{\mathrm{d}t} = \frac{\sin(\frac{\pi}{2}) - \sin(0)}{1-0} = 1.$$

The equation of the trajectory of the shock is $x_s(t) = t$. Finally

$$\rho(x,t) = \begin{cases} 0 & \text{if } x < t, \\ 1 & \text{if } t < x. \end{cases} \tag{5}$$



Question 6: Consider the conservation equation with flux $q(\rho) = \rho^3$. Assume that the initial data is $\rho_0(x) = 2$, if $x < 0$, $\rho_0(x) = 1$, if $0 < x < 1$, and $\rho_0(x) = 0$, if $1 < x$. (i) Draw the characteristics

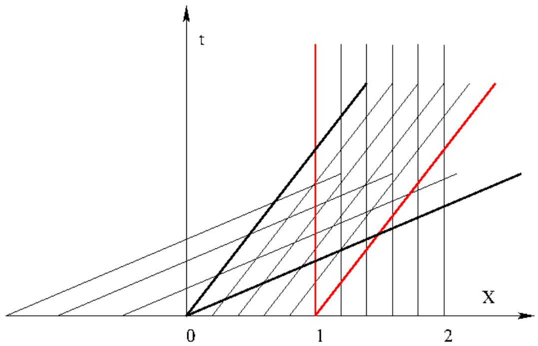
Solution: There are three families of characteristics.

Case 1: $s < 0$, $X(s,t) = 12t + s$. In the x-t plane, these are lines with slope $\frac{1}{12}$.

Case 2: $0 < s < 1$, $X(s,t) = 3t + s$. In the x-t plane, these are lines with slope $\frac{1}{3}$.

Case 3: $1 < s$, $X(s,t) = s$. In the x-t plane, these are vertical lines.

One shock forms between the two black characteristics and another forms between the two red characteristic (see figure).



(ii) Describe qualitatively the nature of the solution.

Solution: We have two shocks moving to the right. One shock forms between the two black characteristics and another forms between the two red characteristic (see figure).

(iii) When and where does the left shock catch up with the right one?

Solution: The speeds of the shocks are

$$\frac{\mathrm{d}x_1(t)}{\mathrm{d}t} = \frac{2^3 - 1}{2 - 1} = 7, \quad \text{and} \quad \frac{\mathrm{d}x_2(t)}{\mathrm{d}t} = \frac{1 - 0}{1 - 0} = 1.$$

The location of the left shock at time t is $x_1(t) = 7t$ and that of the right shock is $x_2(t) = t + 1$. The two shocks are at the same location when $7t = t + 1$, i.e., $t = \frac{1}{6}$; the two shocks merge at $x = \frac{7}{6}$.

(iv) What is the speed of the shock once the two shocks have merged and what is the position of the shock as a function of time?

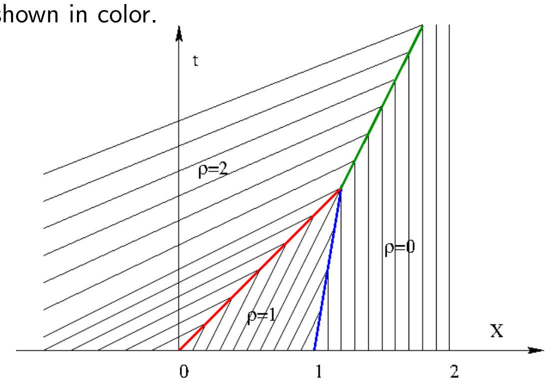
Solution: When the shocks have merged the left state is $\rho = 2$ and the right state is $\rho = 0$; as a result the speed of the shock is

$$\frac{\mathrm{d}x_3(t)}{\mathrm{d}t} = \frac{2^3 - 0}{2 - 0} = 4,$$

and the shock trajectory is $x_3(t) = 4(t - \frac{1}{6}) + \frac{7}{6}$.

(v) Draw precisely all the characteristics of the solution.

Solution: The three shocks are shown in color.



Question 7: Consider the following problem: Let f be a smooth function in $[0, \pi]$, find u such that $u + \partial_{xx} u = f(x)$, $x \in (0, \pi)$, $u(\pi) = 0$, $u(0) = 0$. (i) Under which condition does this problem have a solution.

Solution: Let us compute the null space of the operator $L : \{v \in \mathcal{C}^2(0, \pi) : \ v(\pi) = 0, \ v(0) = 0\} \ni u \longmapsto u(x) + \partial_{xx} u(x) \in \mathcal{C}^0(0, \pi)$. Let $N(L)$ be the null space. Let $v \in N(L)$, then

$$v + \partial_{xx} v = 0$$

which means that $v = a\cos(x) + b\sin(x)$. The boundary conditions imply that $a = 0$; as a result $N(L) = \text{span}(\sin(x))$, i.e., $N(L)$ is the one-dimensional vector space spanned by the function $\sin(x)$. Fredholm's alternative implies that the above problem has a solution only if $\int_0^\pi \sin(x) f(x) \mathrm{d}x = 0$.

(ii) Does the above problem have a solution for $f(x) = \cos(x)$?

Solution: We need to compute $\int_0^\pi \sin(x)\cos(x)\mathrm{d}x$,

$$\int_0^\pi \sin(x)\cos(x)\mathrm{d}x = \frac{1}{2}\int_0^\pi \sin(2x)\mathrm{d}x = 0.$$

The Fredholm alternative implies that the problem has a solution.