Quiz 1 (Notes, books, and calculators are not authorized)

Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with **no justification will not be graded**.

Question 1: Let $\phi(x,y) = \cos(x + \sin(x-y))$. Compute $\partial_x\phi(x,y)$ and $\partial_y\phi(x,y)$. (Do not try to simplify the results).

We apply the chain rule repeatedly

$$\partial_x\phi(x,y) = -\sin(x + \sin(x-y))(1 + \cos(x-y)).$$

$$\partial_y\phi(x,y) = -\sin(x + \sin(x-y))(-\cos(x-y)).$$

Question 2: Consider the heat equation $\partial_t T - k\partial_{xx}T = f(x)$, $x \in [a,b]$, $t > 0$, with $f(x) = kx$, where $k > 0$. Compute the steady state solution (i.e., $\partial_t T = 0$) assuming the boundary conditions: $-k\partial_n T(a) = 0$, $T(b) = 0$ (∂_n is the normal derivative)

At steady state, T does not depend on t and we have $\partial_{xx}T(x) = -x$, which implies $\partial_x T(x) = \alpha - \frac{1}{2}x^2$, and $T(x) = \beta + \alpha x - \frac{1}{6}x^3$, where $\alpha, \beta \in \mathbb{R}$. The two constants α and β are determined by the boundary conditions. $0 = -\partial_n T(a) = \partial_x T(a) = \alpha - \frac{1}{2}a^2$ and $0 = T(b) = \beta + \alpha b - \frac{1}{6}b^3$. We conclude that $\alpha = \frac{1}{2}a^2$ and $\beta = -\alpha b + \frac{1}{6}b^3 = -\frac{1}{2}a^2 b + \frac{1}{6}b^3$. In conclusion

$$T(x) = -\frac{1}{2}a^2 b + \frac{1}{6}b^3 + \frac{1}{2}a^2 x - \frac{1}{6}x^3 = -\frac{1}{2}a^2(b-x) + \frac{1}{6}(b^3 - x^3).$$

Question 3: Consider the equation $\partial_t c(x,t) - \partial_{xx} c(x,t) = x$, where $x \in [0,L]$, $t > 0$, with $c(x,0) = f(x)$, $-\partial_n c(0,t) = 6$, $-\partial_n c(L,t) = 5$, (∂_n is the normal derivative). Compute $E(t) := \int_0^L c(\xi,t)\mathsf{d}\xi$.

We integrate the equation with respect to x over $[0,L]$

$$\int_0^L \partial_t c(\xi,t)\mathsf{d}\xi - \int_0^L \partial_{\xi\xi} c(\xi,t)\mathsf{d}\xi = \int_0^L \xi \mathsf{d}\xi.$$

Using that $\int_0^L \partial_t c(\xi,t)\mathsf{d}\xi = \mathsf{d}_t \int_0^T c(\xi,t)\mathsf{d}\xi$ together with the fundamental theorem of calculus, we infer that

$$\mathsf{d}_t E(t) - \partial_x c(L,t) + \partial_x c(0,t) = \frac{1}{2}L^2.$$

The boundary conditions $\partial_x c(0,t) = -\partial_n c(0,t) = 6$, $-\partial_x c(L,t) = -\partial_n c(L,t) = 5$ give

$$\mathsf{d}_t E(t) + 5 + 6 = \frac{1}{2}L^2.$$

We now apply the fundamental theorem of calculus with respect to t

$$E(t) - E(0) = \int_0^t \partial_\tau E(\tau)\mathsf{d}\tau = (\frac{1}{2}L^2 - 11)t.$$

In conclusion

$$E(t) = \int_0^L f(\xi)\mathsf{d}\xi + (\frac{1}{2}L^2 - 11)t.$$

Question 4: Let $\phi = x^4 - y^4$ (a) Compute $\Delta\phi(x,y)$. (b) Consider the square $\Omega = [0,1]\times[0,1]$ and let Γ be the boundary of Ω. Compute $\int_\Gamma \partial_n \phi \mathrm{d}\Gamma$.

(a) The definition $\Delta\phi = \partial_{xx}\phi + \partial_{yy}\phi$ implies that

$$\Delta\phi = \partial_{xx}\phi + \partial_{yy}\phi = 12x^2 - 12y^2 = 12(x^2 - y^2).$$

(b)The definition $\Delta\phi = \mathsf{div}(\nabla\phi)$ and the fundamental theorem of calculus (also known as the divergence theorem) imply that

$$\int_\Gamma \partial_n\phi \mathsf{d}\Gamma = \int_\Gamma n\cdot\nabla\phi \mathsf{d}\Gamma = \int_\Omega \mathsf{div}(\nabla\phi)\mathsf{d}\Omega = \int_\Omega \Delta\phi \mathsf{d}\Omega = \int_\Omega 12(x^2 - y^2)\mathsf{d}x\mathsf{d}y = 0.$$
