Quizz 2 (Notes, books, and calculators are not authorized)

Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with **no justification will not be graded**.

Question 1: Let ϕ be a non-zero solution to the eigenvalue problem $-\partial_{xx}\phi(x) = \lambda\phi(x)$, $x \in (0,\pi)$, $\phi(0) = 0$, $\phi(\pi) = 0$. Determine the sign of λ using the energy method.

Multiply the equation by ϕ, integrate over $(0,\pi)$, and apply the fundamental theorem of calculus (i.e. integrate by parts):

$$-\int_0^\pi \partial_{xx}\phi(x)\mathsf{d}x = -\int_0^\pi (\partial_x(\phi(x)\partial_x\phi(x)) - (\partial_x\phi(x))^2)\mathsf{d}x$$
$$= -\phi(\pi)\partial_x\phi(\pi) + \phi(0)\partial_x\phi(0) + \int_0^\pi (\partial_x\phi(x))^2\mathsf{d}x = \lambda\int_0^\pi (\phi(x))^2\mathsf{d}x.$$

In conclusion

$$\int_0^\pi (\partial_x\phi(x))^2\mathsf{d}x = \lambda\int_0^\pi (\phi(x))^2\mathsf{d}x.$$

Assuming that ϕ is nonzero, we obtain that $\lambda = \int_0^\pi (\partial_x\phi(x))^2\mathsf{d}x/\lambda\int_0^\pi (\phi(x))^2\mathsf{d}x \geq 0$, i.e. λ is non-negative. If $\lambda = 0$ then $\partial_x\phi = 0$, which implies that ϕ is constant. The boundary conditions imply that $\phi = 0$ which contradicts our assumption that ϕ is non-zero. In conclusion λ is negative.

Question 2: Compute all the positive eigenvalues λ to the above eigenvalue problem.

Owing to λ being non-negative, the general solution to $-\partial_{xx}\phi(x) = \lambda\phi(x)$ is $\phi(x) = a\cos(\sqrt{\lambda}x) + b\sin(\sqrt{\lambda}x)$. The boundary conditions imply that $\phi(0) = 0 = a$ and $\phi(\pi) = b\sin(\sqrt{\lambda}\pi) = 0$. As a result, $\sqrt{\lambda}\pi = n\pi$, where $n \in \mathbb{N}$, i.e.

$$\lambda = n^2, \qquad n \in \mathbb{N}.$$

Question 3: Consider the heat equation $\partial_t u(x,t) - 3\partial_{xx}u(x,t) = 0$, $\partial_x u(0,t) = 0$, $\partial_x u(1,t) = 0$, $u(x,0) = u_0(x)$, $t > 0$, $x \in (0,1)$. The general solution is $u(x,t) = \sum_{n=0}^{\infty} A_n \cos(n\pi x)\mathsf{e}^{-3n^2\pi^2 t}$. Compute the solution corresponding to the initial data $u_0(x) = 5\cos(4\pi x)$.

The solution contains one term only, corresponding to $n = 4$,

$$u(x,t) = 5\cos(4\pi x)\mathsf{e}^{-48\pi^2 t}.$$

Question 4: Assume that the following equation has a smooth solution: $-\partial_x((1+x^2)\partial_x T(x)) + \partial_x T(x) + T(x) = 2x - 1$, $T(a) = 1$, $T(b) = \pi$, $x \in [a,b]$, $t > 0$, where $k > 0$. Prove that this solution is unique by using the energy method. (Hint: Do not try to simplify $-\partial_x((1+x^2)\partial_x T)$.

Assume that there are two solutions T_1 and T_2. Let $\phi = T_2 - T_1$. Then

$$-\partial_x((1+x^2)\partial_x\phi(x)) + \partial_x\phi(x) + \phi(x) = 0, \quad \phi(a) = 0, \quad \phi(b) = 0$$

Multiply the PDE by ϕ, integrate over (a,b), and integrate by parts (i.e. apply the fundamental theorem of calculus):

$$\begin{aligned}
0 &= \int_a^b \big(-\partial_x((1+x^2)\partial_x\phi(x))\phi(x) + (\partial_x\phi(x))\phi(x) + (\phi(x))^2\big)\mathsf{d}x \\
&= \int_a^b \big(-\partial_x(\phi(x)(1+x^2)\partial_x\phi(x)) + (1+x^2)(\partial_x\phi(x))^2 + \partial_x(\frac{1}{2}\phi(x)^2) + (\phi(x))^2\big)\mathsf{d}x \\
&= \int_a^b \big((1+x^2)(\partial_x\phi(x))^2 + (\phi(x))^2\big)\mathsf{d}x
\end{aligned}$$

This implies $\int_a^b (\phi(x))^2\mathsf{d}x = 0$, i.e. $\phi = 0$, meaning that $T_2 = T_1$.