Quiz 3 (Notes, books, and calculators are not authorized)

Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with **no justification will not be graded**.

Question 1: Does any of the following expressions solve the Laplace equation inside the rectangle $0 \le x \le L$, $0 \le y \le H$, with the following boundary conditions $\partial_x u(0,y) = \frac{15\pi}{H}\sin(\frac{5\pi y}{H})\cosh(\frac{5\pi L}{H})$, $\partial_x u(L,y) = 0$, $u(x,0) = 0$, $u(x,H) = 0$? (justify clearly your answer):

$$u_1(x,y) = 3\cos(\frac{5\pi y}{H})\cosh(\frac{5\pi(x-L)}{H}), \quad u_2(x,y) = 3\sin(\frac{5\pi y}{H})\cosh(\frac{5\pi(x-L)}{H}),$$
$$u_3(x,y) = 3\cos(\frac{5\pi y}{H})\sinh(\frac{5\pi(x-L)}{H}), \quad u_4(x,y) = 3\sin(\frac{5\pi y}{H})\sinh(\frac{5\pi(x-L)}{H}).$$

From class, we know that all the above expressions solve the Laplace equation, hence we just need to verify the boundary conditions. We observe that u_1 and u_3 do not satisfy the Dirichlet boundary conditions $u(x,0) = 0$, $u(x,H) = 0$; therefore u_1 and u_3 must be discarded.

Both u_2 and u_4 satify that Dirichlet conditions: $u_2(x,0) = 0$, $u_2(x,H) = 0$, and $u_4(x,0) = 0$, $u_4(x,H) = 0$. Now we need to check the Neumann conditions.

Note that u_4 is such that $\partial_x u_4(L,y) = 3\frac{5\pi}{H}\sin(\frac{5\pi y}{H})\cosh(0) \neq 0$; a result u_4 must be discarded as well.

Finally u_2 is such that $\partial_x u_2(L,y) = 3\frac{5\pi}{H}\sin(\frac{5\pi y}{H})\sinh(0) = 0$, but $\partial_x u_2(0,y) = 3\frac{5\pi}{H}\sin(\frac{5\pi y}{H})\sinh(-\frac{5\pi L}{H})$, which shows that u_2 is not the solution to our problem either.

In conclusion, none of the proposed solutions solve the problem. The correction solution is

$$u(x,y) = 3\frac{\cosh(\frac{5\pi L}{H})}{\sinh(\frac{-5\pi L}{H})}\sin(\frac{5\pi y}{H})\cosh(\frac{5\pi(x-L)}{H}).$$

Question 2: The solution of the equation, $\frac{1}{r}\partial_r(r\partial_r u) + \frac{1}{r^2}\partial_{\theta\theta}u = 0$, inside the domain $D = \{\theta \in [0,\pi],\ r \in [0,2]\}$, subject to the boundary conditions $u(r,0) = 0$, $u(r,\pi) = 0$, $u(2,\theta) = g(\theta)$ is $u(r,\theta) = \sum_{n=1}^{\infty} b_n r^n \sin(n\theta)$. What is the solution corresponding to $g(\theta) = 5\sin(2\theta) + 2\sin(5\theta)$? (Give all the details.)

The only non-zero terms in the expansion are $a_2 r^2 \sin(2\theta) + b_5 r^5 \sin(5\theta)$. The boundary condition $u(2,\theta) = 5\sin(2\theta) + 2\sin(5\theta) = a_2 2^2 \sin(2\theta) + a_5 2^5 \sin(5\theta)$ is satisfied if $a_2 = 5/(2^2)$ and $b_5 = 2/(2^5)$, i.e.,

$$u(r,\theta) = 5\frac{r^2}{2^2}\sin(2\theta) + 2\frac{r^5}{2^5}\sin(5\theta).$$

Question 3: Consider the square $D = (-1,+1)\times(-1,+1)$. Let $f(x,y) = x^2 - y^2 - 3$. Let $u \in \mathcal{C}^2(D) \cap \mathcal{C}^0(\overline{D})$ solve $-\Delta u = 0$ in D and $u|_{\partial D} = f$. Compute $\min_{(x,y)\in\overline{D}} u(x,y)$ and $\max_{(x,y)\in\overline{D}} u(x,y)$.

We use the maximum principle (u is harmonic and has the required regularity). Then

$$\min_{(x,y)\in\overline{D}} u(x,y) = \min_{(x,y)\in\partial D} f(x,y), \quad \text{and} \quad \max_{(x,y)\in\overline{D}} u(x,y) = \max_{(x,y)\in\partial D} f(x,y).$$

A point (x,y) is at the boundary of D if and only if $x^2 = 1$ and $y \in (-1,1)$ or $y^2 = 1$ and $x \in (-1,1)$. In the first case, $x^2 = 1$ and $y \in (-1,1)$, we have

$$f(x,y) = 1 - y^2 - 3, \qquad y \in (-1,1).$$

The maximum is -2 and the minimum is -3. In the second case, $y^2 = 1$ and $x \in (-1,1)$, we have

$$f(x,y) = x^2 - 1 - 3, \qquad x \in (-1,1).$$

The maximum is -3 and the minimum is -4. We finally can conclude

$$\min_{(x,y)\in\partial D} f(x,y) = \min_{-1\le x\le 1} x^2 - 4, = -4, \quad \max_{(x,y)\in\partial D} f(x,y) = \max_{-1\le y\le 1} -2 - y^2 = -2.$$

In conclusion

$$\min_{(x,y)\in\overline{D}} u(x,y) = -4, \quad \max_{(x,y)\in\overline{D}} u(x,y) = -2$$
