Quiz 7 (Notes, books, and calculators are not authorized)

Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with **no justification will not be graded**.

Question 1: Consider the equation $-\partial_x(x\partial_x u(x)) = f(x)$ for all $x \in (1,2)$ with $u(1) = a$ and $u(2) = b$. Let $G(x, x_0)$ be the associated Green's function.
(i) Use the Fredholm alternative to determine whether the above problem has a unique solution when $a = b = 0$.

Let us prove that the null space of the operator $L : \{v \in \mathcal{C}^2(1,2) : \ v(1) = 0, \ v(2) = 0\} \ni u \longmapsto \partial_x(x\partial_x u(x)) \in \mathcal{C}^0(1,2)$ reduces to $\{0\}$. Let $N(L)$ be the null space. Let $v \in N(L)$. By using the usual energy argument, we infer that

$$\int_0^1 x(\partial_x v)^2 \mathsf{d}x = 0.$$

This means that v is constant over the interval $(1,2)$, but the boundary condition $v(1) = 0$ implies that this constant is zero, thereby proving that $v = 0$. In conclusion $N(L) = \{0\}$. Fredholm's alternative implies that the above problem has a unique solution.

(ii) Give the equation and boundary conditions satisfied by G and give the integral representation of $u(x_0)$ for all $x_0 \in (1,2)$ in terms of G, f, and the boundary data. (Do not compute G in this question).

We have a second-order PDE and the operator is clearly self-adjoint. The Green's function solves the equation

$$-\partial_x(x\partial_x G(x, x_0)) = \delta(x - x_0), \quad G(1, x_0) = 0, \quad G(2, x_0) = 0.$$

We multiply the equation by u and integrate over the domain $(1,2)$ (in the distribution sense).

$$\langle \delta(x - x_0), u \rangle = u(x_0) = -\int_1^2 \partial_x(x\partial_x G(x, x_0))u(x)\mathsf{d}x.$$

We integrate by parts and we obtain,

$$\begin{aligned} u(x_0) &= \int_1^2 x\partial_x G(x, x_0)\partial_x u(x)\mathsf{d}x - [x\partial_x G(x, x_0)u(x)]_1^2 \\ &= -\int_1^2 G(x, x_0)\partial_x(x\partial_x u(x))\mathsf{d}x - 2\partial_x G(2, x_0)u(2) + \partial_x G(1, x_0)u(1). \end{aligned}$$

Now, using the boundary conditions and the fact that $-\partial_x(x\partial_x u(x)) = f(x)$, we finally have

$$u(x_0) = \int_1^2 G(x, x_0)f(x)\mathsf{d}x - 2\partial_x G(2, x_0)b + \partial_x G(1, x_0)a.$$

(iii) Compute $G(x,x_0)$ for all $x,x_0\in(1,2)$.

For all $x\neq x_0$ we have

$$-\partial_x(x\partial_x G(x,x_0))=0.$$

The solution is

$$G(x,x_0)=\begin{cases} a\log(x)+b & \text{if } 1<x<x_0\\ c\log(x)+d & \text{if } x_0<x<2\end{cases}$$

The boundary conditions give $b=0$ and $d=-c\log(2)$; as a result,

$$G(x,x_0)=\begin{cases} a\log(x) & \text{if } 1<x<x_0\\ c\log(x/2) & \text{if } x_0<x<2\end{cases}$$

G must be continuous at x_0,

$$a\log(x_0)=c\log(x_0)-c\log(2)$$

and must satisfy the gap condition

$$-\int_{x_0-\epsilon}^{x_0+\epsilon}\partial_x(x\partial_x G(x,x_0))\mathrm{d}x=1,\qquad \forall\epsilon>0.$$

This gives

$$-x_0\left(\partial_x G(x_0^+,x_0)-G(x_0^-,x_0)\right)=1$$
$$-x_0\left(\frac{c}{x_0}-\frac{a}{x_0}\right)=1$$

This gives

$$a-c=1.$$

In conclusion $\log(x_0)=-c\log 2$ and

$$c=-\log(x_0)/\log(2),\quad a=1-\log(x_0)/\log(2)=\log(2/x_0)/\log(2).$$

This means

$$G(x,x_0)=\begin{cases} \frac{\log(2/x_0)}{\log(2)}\log(x) & \text{if } 1<x<x_0\\ \frac{\log(x_0)}{\log(2)}\log(2/x) & \text{if } x_0<x<2\end{cases}$$

Question 2: Use the Fredholm alternative to determine whether the following problem has a unique solution: Let f be a smooth function in $[0,1]$, find u such that

$$u-\partial_{xx}u=f(x),\quad x\in(0,1),\qquad \partial_x u(1)+u(1)=0,\ -\partial_x u(0)+u(0)=0$$

Fully justify your answer.

Let us compute the null space of the operator $L:\{v\in\mathcal{C}^2(0,1):\ \partial_x v(1)+v(1)=0,\ -\partial_x v(0)+v(0)=0\}\ni u\longmapsto u(x)-\partial_{xx}u(x)\in\mathcal{C}^0(0,1)$. Let $N(L)$ be the null space. Let $v\in N(L)$. The energy argument implies that

$$0=\int_0^1(v^2+(\partial_x v)^2)\mathrm{d}x-[v\partial_x v]_0^1=\int_0^1(v^2+(\partial_x v)^2)\mathrm{d}x+v(1)^2+v(0)^2.$$

This proves that $v=0$, i.e., $N(L)=\{0\}$. Fredholm's alternative implies that the above problem has a unique solution.