Quiz 8 (Notes, books, and calculators are not authorized)

Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with **no justification will not be graded**.

The implicit representation of the solution to the equation $\partial_t v + \partial_x q(v) = 0$, $v(x,0) = v_0(x)$, is

$$X(s,t) = q'(v_0(s))t + s; \quad v(X(s,t),t) = v_0(s). \tag{1}$$

Question 1: Consider the following conservation equation

$$\partial_t \rho + \partial_x(q(\rho)) = 0, \quad x \in (-\infty, +\infty),\ t > 0, \qquad \rho(x,0) = \rho_0(x) := \begin{cases} \frac{1}{2} & \text{if } x < 0, \\ 1 & \text{if } x > 0, \end{cases}$$

where $q(\rho) = \rho(2 - \sin(\rho))$ (and $\rho(x,t)$ is the conserved quantity). What is the wave speed for this problem?

The wave speed if the quantity $q'(\rho) = 2 - \sin(\rho) - \rho\cos(\rho)$.

Question 2: Consider the following conservation equation

$$\partial_t \rho + \partial_x(q(\rho)) = 0, \quad x \in (-\infty, +\infty),\ t > 0, \qquad \rho(x,0) = \rho_0(x) := \begin{cases} \frac{1}{6} & \text{if } x < 0, \\ \frac{1}{3} & \text{if } x > 0, \end{cases}$$

where $q(\rho) = \rho(2 - 3\rho)$ (and $\rho(x,t)$ is the conserved quantity). (i) Given that this initial data produces a shock, give the speed of the shock.

The Rankin-Hugoniot relation gives the speed of this shock:

$$\frac{\mathrm{d}x_s}{\mathrm{d}t} = \frac{q^+ - q^-}{\rho^+ - \rho^-} = \frac{\frac{1}{6}\frac{3}{2} - \frac{1}{3}}{\frac{1}{6} - \frac{1}{3}} = \frac{1}{12}6 = \frac{1}{2}.$$

(ii) Give the solution to the problem.

In conclusion the location of the shock is $x_s(t) = \frac{1}{2}t$ and the explicit representation of the solution is

$$\rho = \frac{1}{6}, \quad \text{if} \quad x < \frac{t}{2} = x_s(t),$$
$$\rho = \frac{1}{3}, \quad \text{if} \quad x > \frac{t}{2} = x_s(t).$$

Question 3: Solve the conservation equation $\partial_t \rho + \partial_x q(\rho) = 0$, $x \in (\infty, +\infty)$, $t > 0$ with flux $q(\rho) = \rho^4 + 2\rho$, and with the initial condition $\rho(x,0) = 1$, if $x < 0$, $\rho(x,0) = -1$, if $x > 0$. Do we have a shock or an expansion wave here?

The solution is given by the implicit representation

$$\rho(X(s,t),t) = \rho_0(s), \quad X(s,t) = s + (4\rho_0(s)^3 + 2)t.$$

We then have two cases depending whether s is positive or negative.
Case 1: $s < 0$, then $\rho_0(s) = 1$ and $X(s,t) = (4+2)t + s = 6t + s$. This means

$$\rho(x,t) = 1 \quad \text{if} \quad x < 6t.$$

Case 2: $s > 0$, then $\rho_0(s) = -1$ and $X(s,t) = (-4+2)t + s = -2t + s$. This means

$$\rho(x,t) = -1 \quad \text{if} \quad x > -2t.$$

We see that the characteristics cross in the region $\{6t > x > -2t\}$. This implies that there is a shock. The Rankin-Hugoniot relation gives the speed of this shock with $\rho^- = 1$ and $\rho^+ = -1$:

$$\frac{\mathrm{d}x_s(t)}{\mathrm{d}t} = \frac{q^+ - q^-}{\rho^+ - \rho^-} = \frac{-1-3}{-1-1} = 2, \qquad x_s(0) = 0.$$

In conclusion the location of the shock is $x_s(t) = 2t$ and the solution is as follows:

$$\begin{aligned} \rho = 1, &\quad \text{if} \quad x < x_s(t) = 2t, \\ \rho = -1, &\quad \text{if} \quad x > x_s(t) = 2t. \end{aligned}$$
