3 Squares, 2 Cardioids and a Circle: Using GeoGebra to Investigate an Interesting Geometry Problem

Thomas J. Clark

University of Nebraska-Lincoln

November 15, 2012

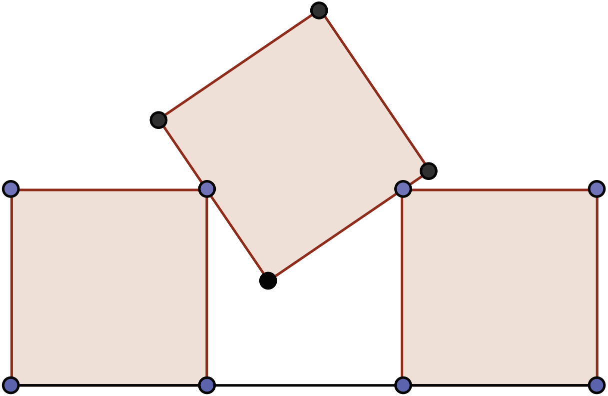
What I'm not Talking about Today:

"I am, and will ever be a white-socks, pocket protector, nerdy engineer, born under the second law of thermodynamics, steeped in steam tables, in love with free-body diagrams, transformed by Laplace and propelled by compressible flow."

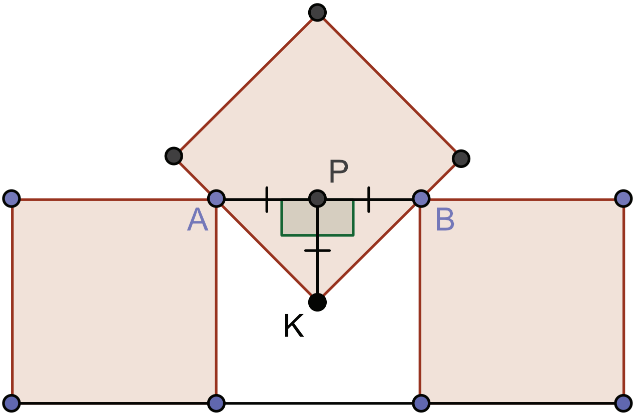
– Neil Armstrong (1930-2012)

How Low Can You Go?

If the top square is allowed to slide, what is the lowest point the bottom corner can reach?



In the Middle



Proof by Path

What is the path of the lowest corner point?

Proof by Path

What is the path of the lowest corner point?

We can use GeoGebra to trace the path...

Proof by Path

What is the path of the lowest corner point?

We can use GeoGebra to trace the path...

Analytic Geometry Proof:

$$(x - .5)^2 + y^2 + (x + .5)^2 + y^2 = 1$$
$$2x^2 + 2y^2 + 1/2 = 1$$
$$x^2 + y^2 = 1/4$$

Geometric Insight Proof:

The line segment $\overline{AB}$ is fixed diameter. The path traced out by point K is the locus of all points that form a right angle with the points A and B.
A circle is the unique curve which satisfies that property, so the path must be a circle.

You Know What They Say About Curiosity

- What about the paths of the other points?

You Know What They Say About Curiosity

- What about the paths of the other points?
- How high does the top corner go?

You Know What They Say About Curiosity

- What about the paths of the other points?
- How high does the top corner go?
- What paths do the left and right corners trace?

You Know What They Say About Curiosity

- What about the paths of the other points?
- How high does the top corner go?
- What paths do the left and right corners trace?

Let's use GeoGebra to investigate these questions.

Some Background on Cardioids

What do you remember about cardioids?

Some Background on Cardioids

What do you remember about cardioids?

- Polar Equation: $r = 2a(1 - \cos\theta)$.
- Traced by a fixed point on a circle, as it rolls around a circle of the same radius.

Some Background on Cardioids

What do you remember about cardioids?

- Polar Equation: $r = 2a(1 - \cos\theta)$.
- Traced by a fixed point on a circle, as it rolls around a circle of the same radius.

Can we connect either one of of these notions of a cardioid to our problem to prove that the corners do in fact trace out cardioids?

Some Background on Cardioids

What do you remember about cardioids?

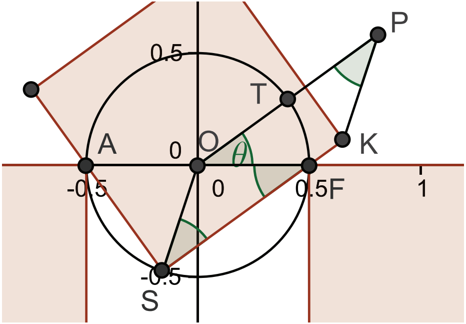
- Polar Equation: $r = 2a(1 - \cos\theta)$.
- Traced by a fixed point on a circle, as it rolls around a circle of the same radius.

Can we connect either one of of these notions of a cardioid to our problem to prove that the corners do in fact trace out cardioids?

In fact we can do both.

Rolling Circles Gather No Moss

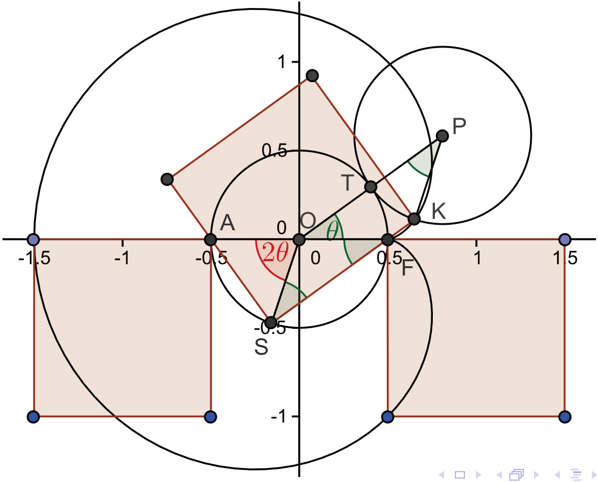
Since there already is a circle in the problem, let's begin by showing that the corner point lies on a circle which rolls around the central circle.



By standard theorems from Geometry, all the marked angles are congruent, and this shows that the circle through K about P is rolling.

Finding the Formula

In the standard formula, $r = 2a(1 - \cos\theta)$, the cusp point must be the origin, so in the diagram that is the point F.



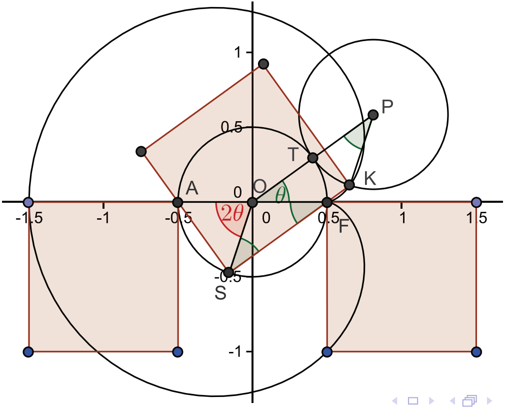
Finding the Formula

Think of F as the 'origin'. From this perspective the point K has polar coordinates (r, θ), where r is to be determined.
Using right triangle trigonometry, the point S is $\frac{1}{2}\cos(2\theta)$ units to the left of O and $\frac{1}{2}\sin(2\theta)$ units down. So with F as the origin, S has Cartesian coordinates $(-\frac{1}{2} - \frac{1}{2}\cos(2\theta), -\frac{1}{2}\sin(2\theta))$. Thus the distance squared d^2 from the origin F to S is

$$\begin{aligned} d^2 &= \Big(-\frac{1}{2} - \frac{1}{2}\cos(2\theta)\Big)^2 + \Big(-\frac{1}{2}\sin(2\theta)\Big)^2 \\ &= \frac{1}{4} + \frac{1}{2}\cos(2\theta) + \frac{1}{4}\cos^2(2\theta) + \frac{1}{4}\sin^2(2\theta) \\ &= \frac{1+\cos(2\theta)}{2} = \cos^2(\theta). \end{aligned}$$

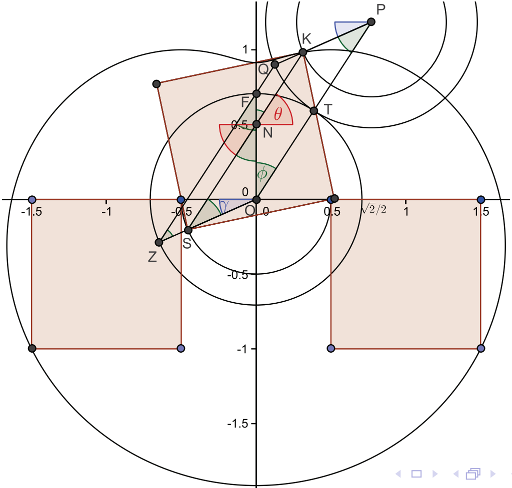
Finding the Formula

Thus $d = |\cos(\theta)|$. In fact for any θ, the directed distance from F to S is $d = -\cos(\theta)$. Thus adding 1, the length of the edge, to d gives us the position of K in direction of θ. That is K has polar coordinates $r = 1 - \cos(\theta)$, the equation of a cardioid.



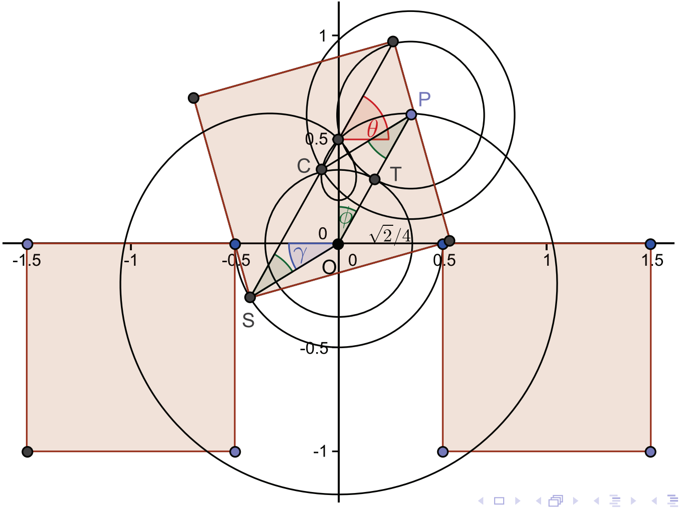
Did You Say Limaçon?

One can similarly prove that the top corner traces out a Limaçon, but I'll spare you the details...



Another Limaçon

The point C at the center of the square also traces out a Limaçon.



Thank You for Coming

- GeoGebra is available free from www.GeoGebra.org